PHYSICS 221A

Practice Final

Real Exam: Tuesday December 11th, 2012, 8am - 11am, PAB 2-434

- Please write clearly
- Print your name on every page used, including this one;
- Make clear which question and which part you are answering on each page.
- No core-dumps please !
- No books, notes, computers, or calculators are allowed during the exam;
- Please turn off all electronic devices.
- All parts of questions, a),b)c) etc., carry a weight of 5 points unless otherwise indicated.

question	possible points	achieved points
1.	20	
2.	20	
3.	20	
4.	15	
5.	20	
Total	95	

Some possibly useful formulas

• The angular momentum algebra is given by $[J_1, J_2] = i\hbar J_3$, and cyclic permutations. The ladder operators, defined by $J_\pm \equiv J_1 \pm iJ_2$, act as follows,

$$J_\pm|j,m\rangle = \hbar\sqrt{j(j+1)-m(m\pm 1)}\,|j,m\pm 1\rangle \tag{0.1}$$

where the states are properly normalized by $\langle j',m'|j,m\rangle = \delta_{j,j'}\delta_{m,m'}$.

• CBH-formula

$$e^A B e^{-A} = \exp(Ad_A)B \tag{0.2}$$

where $Ad_X \cdot = [X,\cdot]$.

• The Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{0.3}$$

• harmonic oscillator

$$\begin{aligned} a &= \frac{1}{\sqrt{2m\omega\hbar}}\Big(m\omega\,X + i\,P\Big) \\ a^\dagger &= \frac{1}{\sqrt{2m\omega\hbar}}\Big(m\omega\,X - i\,P\Big) \end{aligned} \tag{0.4}$$

QUESTION 1: [20 points]

A system of three (non-identical) spin 1/2 particles, whose spin operators are $\vec{S}_1, \vec{S}_2, \vec{S}_3$, is governed by the Hamiltonian,

$$H = \frac{2A}{\hbar^2}\vec{S}_1 \cdot \vec{S}_2 + \frac{2B}{\hbar^2}\vec{S}_3 \cdot (\vec{S}_1 + \vec{S}_2)$$

where A and B are real constants.

a) [5pts] Rewrite the Hamiltonian such that in only involves squares of $\vec{S}_i$ or squares of sums of $\vec{S}_i$'s.

b) [15pts] Calculate the energy levels and their respective degeneracies.

Note: In part b) you can quote results of representation theory and addition of angular momentum

QUESTION 2: [20 points]

A coherent state for a single harmonic oscillator is given by

$$| c\rangle_{\text{coh}} = e^{\frac{-|c|^2}{2}} e^{ca^\dagger} \mid 0\rangle \tag{0.5}$$

Where $\mid 0\rangle$ is the ground state of the harmonic oscillator.

a) Show that $\mid c\rangle_{\text{coh}}$ is normalized

b) Are there any values of c_1, c_2 for which two coherent states are orthogonal ? (Back up your answer with an argument or a calculation).

c) Show that $\mid c\rangle_{coh}$ is an eigenstate of the lowering operator a and calculate the eigenvalue.

d) Show that

$$|\langle n \mid c\rangle_{coh}|^2 = \frac{A^n}{n!} e^{-B} \tag{0.6}$$

and determine A and B. Here

$$\mid n\rangle = \frac{1}{\sqrt{n!}} (a^\dagger)^n \mid 0\rangle \tag{0.7}$$

is the n-th excited state of the harmonic oscillator.

QUESTION 3: [20 points]

Consider a one dimensional particle moving in a potential with Hamiltonian

$$H = \frac{p^2}{2m} + V(x) \tag{0.8}$$

Assume that the Hamiltonian has a discrete non-degenerate spectrum (i.e. there are only bound states)

$$H \mid n\rangle = E_n \mid n\rangle, \quad E_n \neq E_m \text{ if } n \neq m \tag{0.9}$$

a) Show that

$$[[x, H], x] = \frac{\hbar^2}{m} \tag{0.10}$$

b) Show that the following "sum rule" holds (it's called that because you sum over all states).

$$\sum_m (E_m - E_n) \left| \langle n \mid x \mid m\rangle \right|^2 = \frac{\hbar^2}{2m} \tag{0.11}$$

Hint: insert a complete set of states in the appropriate place.

c) Calculate

$$\langle n \mid x \mid m\rangle \tag{0.12}$$

for the harmonic oscillator

d) Verify the sum rule for the harmonic oscillator

QUESTION 4: [15 points]

The states $\mid \psi(t)\rangle$ and $\mid \phi(t)\rangle$ both satisfy the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t} \mid \Psi(t)\rangle = H \mid \Psi(t)\rangle \tag{0.13}$$

Assume that the Hamiltonian H does not have any explicit time dependence. For a),b) work in the Schrödinger picture.

a) Show that any $\mid \Psi(t)\rangle$ which solves (0.13) can be expressed as follows:

$$\mid \Psi(t)\rangle = e^{-\frac{i}{\hbar}Ht} \mid \psi(0)\rangle \tag{0.14}$$

b) At time $t = 0$ the two states are related by

$$\mid \psi(0)\rangle = F \mid \phi(0)\rangle \tag{0.15}$$

What is the condition on the operator F that this relation holds also for later times, i.e.

$$\mid \psi(t)\rangle = F \mid \phi(t)\rangle \tag{0.16}$$

c) If you transform the operator F from the Schrödinger picture to the Heisenberg picture, will it be time dependent ? Back up your answer by a calculation.

QUESTION 5: [20 points]

Consider the one dimensional periodic Ising chain with 3 sites

$$H = -\frac{j}{\hbar^2}\Big(S_1^z S_2^z + S_2^z S_3^z + S_3^z S_1^z\Big) + \frac{b}{\hbar}\Big(S_1^z + S_2^z + S_3^z\Big) \tag{0.17}$$

Where $\vec{S}_i$ is the spin 1/2 operator of the i-th site (treat the spins as distinguishable).

a) Find a normalized basis of eigenstates of H.

b) Find the spectrum of H and its degeneracy

c) For $b \neq 0$ calculate the partition function

$$Z = \mathrm{tr}(e^{-\beta H}) \tag{0.18}$$

d) Calculate the thermal expectation value of $S_1^z + S_2^z + S_3^z$ for the case $b = 0$

$$< S_1^z + S_2^z + S_3^z >= \mathrm{tr}\Big((S_1^z + S_2^z + S_3^z)e^{-\beta H}\Big)\,|_{b=0} \tag{0.19}$$

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where the states are properly normalized by $\langle j', m'|j, m\rangle = \delta_{j,j'}\delta_{m,m'}$.

• CBH-formula

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$$H = \frac{2A}{\hbar^2}\vec{S}_1 \cdot \vec{S}_2 + \frac{2B}{\hbar^2}\vec{S}_3 \cdot (\vec{S}_1 + \vec{S}_2)$$

where A and B are real constants.

a) [5pts] Rewrite the Hamiltonian such that in only involves squares of $\vec{S}_i$ or squares of sums of $\vec{S}_i$'s.

b) [15pts] Calculate the energy levels and their respective degeneracies.

Note: In part b) you can quote results of representation theory and addition of angular momentum

Solution:

a) The Hamiltonian can be expressed as follows:

$$\begin{aligned} H &= \frac{2A}{\hbar^2}\vec{S}_1 \cdot \vec{S}_2 + \frac{2B}{\hbar^2}\vec{S}_3 \cdot (\vec{S}_1 + \vec{S}_2) \\ &= \frac{A}{\hbar^2}((S_1+S_2)^2 - S_1^2 - S_2^2) + \frac{B}{\hbar^2}((S_1+S_2+S_3)^2 - S_3^2 - (S_1+S_2)^2) \end{aligned} \tag{0.5}$$

b) Following the rule of adding angular momentum, the $8 = 2\times 2\times 2$ states of the three fold tensor product decompose as follows, using the rules of addition of momentum.

$$\begin{aligned} D_1^{(1/2)} \otimes D_2^{(1/2)} \otimes D_3^{(1/2)} &= (D_{12}^{(0)} \oplus D_{12}^{(1)}) \otimes D_3^{(1/2)} \\ &= D_{123}^{(1/2)} \oplus 2\, D_{123}^{(1/2)} \oplus D_{123}^{(3/2)} \end{aligned} \tag{0.6}$$

the subscript of D refers to the particle contents of the representation, where we first added the spin of particle 1 and 2 and then afterwards the spin of particle 3. Note that the first $D_{123}^{(1/2)}$ and the second $D_{123}^{(1/2)}$ differ, the first one has $S_1 + S_2 = 0$, whereas the second has $S_1 + S_2 = 1$.

With this decomposition, we can read off the eigenvalues of H.

$$\begin{aligned} E_1 &= \frac{A}{\hbar^2}(0 - 3/4 - 3/4)\hbar^2 + \frac{B}{\hbar^2}(3/4 - 3/4 - 0)\hbar^2 = -\frac{3A}{2}; \qquad S_{12} = 0 \quad S_{123} = \frac{1}{2} \\ E_2 &= \frac{A}{\hbar^2}(2 - 3/4 - 3/4)\hbar^2 + \frac{B}{\hbar^2}(3/4 - 3/4 - 2)\hbar^2 = \frac{A}{2} - 2B; \qquad S_{12} = 1 \quad S_{123} = \frac{1}{2} \\ E_3 &= \frac{A}{\hbar^2}(2 - 3/4 - 3/4)\hbar^2 + \frac{B}{\hbar^2}(15/4 - 3/4 - 2)\hbar^2 = \frac{A}{2} + B; \qquad S_{12} = 1 \quad S_{123} = \frac{3}{2} \end{aligned} \tag{0.7}$$

So the spectrum and degeneracies are

$$\begin{array}{cc} E & \text{deg.} \\ -\frac{3A}{2} & 2 \\ \frac{A}{2} - 2B & 2 \\ \frac{A}{2} + B & 4 \end{array} \tag{0.8}$$

QUESTION 2: [20 points]

A coherent state for a single harmonic oscillator is given by

$$\mid c\rangle_{\rm coh} = e^{\frac{-|c|^2}{2}} e^{ca^\dagger} \mid 0\rangle \tag{0.9}$$

Where $\mid 0\rangle$ is the ground state of the harmonic oscillator.

a) Show that $\mid c\rangle_{\rm coh}$ is normalized

b) Are there any values of c_1, c_2 for which two coherent states are orthogonal ? (Back up your answer with an argument or a calculation).

c) Show that $\mid c\rangle_{coh}$ is an eigenstate of the lowering operator a and calculate the eigenvalue.

d) Show that

$$|\langle n \mid c\rangle_{coh}|^2 = \frac{A^n}{n!} e^{-B} \tag{0.10}$$

and determine A and B. Here

$$\mid n\rangle = \frac{1}{\sqrt{n!}} (a^\dagger)^n \mid 0\rangle \tag{0.11}$$

is the n-th excited state of the harmonic oscillator.

Solution:

a) One has

$$\begin{aligned} || \mid c\rangle_{\rm coh}||^2 &= {}_{\rm coh}\langle c \mid c\rangle_{\rm coh} \\ &= e^{-|c|^2} \langle 0 \mid e^{c^* a} e^{ca^\dagger} \mid 0\rangle \\ &= e^{-|c|^2} \langle 0 \mid e^{ca^\dagger} e^{c^* a} e^{|c|^2 [a, a^\dagger]} \mid 0\rangle \\ &= e^{-|c|^2} \langle 0 \mid e^{|c|^2} \mid 0\rangle \\ &= 1 \end{aligned} \tag{0.12}$$

In the third line we have used

$$e^{ca} \mid 0\rangle = \mid 0\rangle, \quad \langle 0 \mid e^{ca^\dagger} = \langle 0 \mid \tag{0.13}$$

and the BCH formula

$$e^X e^Y = e^{X+Y} e^{\frac{1}{2}[X,Y]} \tag{0.14}$$

which holds if $[X, Y]$ is a c-number. It follows

$$e^X e^Y = e^Y e^X e^{[X,Y]} \tag{0.15}$$

b) One has

$$\begin{aligned}
{}_{\rm coh}\langle c_1 \mid c_2\rangle_{\rm coh} &= e^{-\frac{1}{2}|c_1|^2-\frac{1}{2}|c_2|^2}\langle 0 \mid e^{c_1^* a} e^{c_2 a^\dagger} \mid 0\rangle \\
&= e\langle 0 \mid e^{c_2 a^\dagger} e^{c_1^* a} e^{c_2 c_1^*[a,a^\dagger]} \mid 0\rangle \\
&= e^{-\frac{1}{2}|c_1|^2-\frac{1}{2}|c_2|^2-c_2 c_1^*}\langle 0 \mid 0\rangle \\
&= e^{-\frac{1}{2}|c_1|^2-\frac{1}{2}|c_2|^2-c_2 c_1^*}
\end{aligned} \tag{0.16}$$

The exponent is never zero, unless one takes the limit $|c_1|$ or $|c_2|$ going to infinity, hence the coherent states for to values of c are never orthogonal.

c) We can use the BCH formula

$$e^A B e^{-A} = \exp(Ad_A)B \tag{0.17}$$

with $A = -ca^\dagger$ and $B = a$. Using the fact that $[a^\dagger, a] = -1$ is a c-number and all higher commutators in $exp(Ad)$ vanish one gets

$$\begin{aligned}
e^{-ca^\dagger} a e^{ca^\dagger} &= a + [-ca^\dagger, a] \\
&= a + c
\end{aligned} \tag{0.18}$$

And hence has

$$a e^{ca^\dagger} = e^{ca^\dagger} a + c e^{ca^\dagger} \mid 0\rangle \tag{0.19}$$

Using this one gets

$$\begin{aligned}
a \mid c\rangle_{\rm coh} &= e^{\frac{-|c|^2}{2}} a e^{ca^\dagger} \mid 0\rangle \\
&= c e^{\frac{-|c|^2}{2}} e^{ca^\dagger} \mid 0\rangle + e^{\frac{-|c|^2}{2}} e^{ca^\dagger} a \mid 0\rangle \\
&= c e^{\frac{-|c|^2}{2}} e^{ca^\dagger} \mid 0\rangle \\
&= c \mid c\rangle_{\rm coh}
\end{aligned} \tag{0.20}$$

The eigenvalue is c.

d) One uses that

$$\langle n \mid = \frac{1}{\sqrt{n!}}\langle 0 \mid a^n \tag{0.21}$$

Hence

$$\begin{aligned}
\langle n \mid\mid c\rangle_{\rm coh} &= \frac{1}{\sqrt{n!}}\langle 0 \mid a^n \mid c\rangle_{\rm coh} \\
&= \frac{1}{\sqrt{n!}} c^n \langle 0 \mid c\rangle_{\rm coh} \\
&= \frac{1}{\sqrt{n!}} c^n e^{\frac{-|c|^2}{2}}
\end{aligned} \tag{0.22}$$

For the absolute value squared one obtains

$$|\langle n \mid c\rangle_{coh}|^2 = \frac{|c|^{2n}}{n!} e^{-|c|^2} \tag{0.23}$$

and hence

$$A = |c|^2, \quad B = |c|^2 \tag{0.24}$$

QUESTION 3: [20 points]

Consider a one dimensional particle moving in a potential with Hamiltonian

$$H = \frac{p^2}{2m} + V(x) \tag{0.25}$$

Assume that the Hamiltonian has a discrete non-degenerate spectrum (i.e. there are only bound states)

$$H \mid n\rangle = E_n \mid n\rangle, \quad E_n \neq E_m \text{ if } n \neq m \tag{0.26}$$

a) Show that

$$[[x, H], x] = \frac{\hbar^2}{m} \tag{0.27}$$

b) Show that

$$\sum_m (E_m - E_n) \left| \langle n \mid x \mid m\rangle \right|^2 = \frac{\hbar^2}{2m} \tag{0.28}$$

Hint: insert a complete set of states in the appropriate place.

c) Calculate

$$\langle n \mid x \mid m\rangle \tag{0.29}$$

for the harmonic oscillator

d) Verify the sum rule for the harmonic oscillator

Solution:

a) We first calculate the commutator

$$[x, H] = \frac{1}{2m}[x, p^2] = \frac{1}{2m}([x, p]p + p[x, p]) = \frac{i\hbar}{m}p \tag{0.30}$$

and from this the double commutator

$$[[x, H], x] = \frac{i\hbar}{m}[p, x] = \frac{\hbar^2}{m} \tag{0.31}$$

b) The expectation value of the commutator for a eigen state $\mid n\rangle$ is then given by

$$\langle n \mid [[x, H], x] \mid n\rangle = \frac{\hbar^2}{m}\langle n \mid n\rangle = \frac{\hbar^2}{m} \tag{0.32}$$

Next insert a complete set of states $\sum_{n'} \mid n'\rangle\langle n' \mid$ in the outer commutator

$$\begin{aligned}
\langle n \mid [[x,H],x] \mid n\rangle &= \sum_{n'} \Big(\langle n \mid [x,H] \mid n'\rangle\langle n' \mid x \mid n\rangle - \langle n \mid x \mid n'\rangle\langle n' \mid [x,H] \mid n\rangle\Big) \\
&= \sum_{n'} \Big((E_{n'} - E_n)|\langle n|x|n'\rangle|^2 - (E_n - E_{n'})|\langle n|x|n'\rangle|^2\Big) \\
&= 2\sum_{n'} \Big((E_{n'} - E_n)|\langle n|x|n'\rangle|^2 \qquad (0.33)
\end{aligned}$$

Equating (0.30) and (0.31) and dividing by 2 proves the sum rule.

c) One has

$$a = \sqrt{\frac{m\omega}{2\hbar}}x + \frac{i}{2m\omega\hbar}p, \quad a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}x - \frac{i}{2m\omega\hbar}p \qquad (0.34)$$

and

$$\mid n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n \mid 0\rangle \qquad (0.35)$$

Hence

$$\begin{aligned}
\langle n \mid x \mid n'\rangle &= \sqrt{\frac{\hbar}{2m\omega}}\frac{1}{\sqrt{n!}}\frac{1}{\sqrt{n'!}}\langle 0 \mid a^n(a+a^\dagger)(a^\dagger)^{n'} \mid 0\rangle \\
&= \sqrt{\frac{\hbar}{2m\omega}}\frac{1}{\sqrt{n!}}\frac{1}{\sqrt{n'!}}\Big(\langle 0 \mid a^{n+1}(a^\dagger)^{n'} \mid 0\rangle + \Big(\langle 0 \mid a^n(a^\dagger)^{n'+1} \mid 0\rangle\Big) \\
&= \sqrt{\frac{\hbar}{2m\omega}}\frac{1}{\sqrt{n!}}\frac{1}{\sqrt{n'!}}\Big((n+1)!\delta_{n+1,n'} + (n'+1)!\delta_{n,n'+1}\Big) \\
&= \sqrt{\frac{\hbar}{2m\omega}}\Big(\frac{(n+1)!}{\sqrt{n!(n+1)!}}\delta_{n+1,n'} + \frac{(n'+1)!}{\sqrt{(n'+1)!n'!}}\delta_{n,n'+1}\Big) \\
&= \sqrt{\frac{\hbar}{2m\omega}}\Big(\sqrt{n+1}\delta_{n+1,n'} + \sqrt{n'+1}\delta_{n,n'+1}\Big) \qquad (0.36)
\end{aligned}$$

When one squares the matrix element the cross terms vanish since both conditions in the Kronecker deltas cannot be satisfied at the same time hence using

$$E_{n'} - E_n = \hbar\omega(n' - n) \qquad (0.37)$$

one gets for (0.31):

$$\begin{aligned}
2\hbar\omega\sum_{n'}(n'-n)\frac{\hbar}{2m\omega}\Big((n+1)\delta_{n',n+1} + n\delta_{n'+1,n}\Big) &= \frac{\hbar^2}{m}(n+1-n) \\
&= \frac{\hbar^2}{m} \qquad (0.38)
\end{aligned}$$

Hence the sum rule holds for the harmonic oscillator.

QUESTION 4: [15 points]

The states $\mid \psi(t)\rangle$ and $\mid \phi(t)\rangle$ both satisfy the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t} \mid \Psi(t)\rangle = H \mid \Psi(t)\rangle \tag{0.39}$$

Assume that the Hamiltonian H does not have any explicit time dependence. For a),b) work in the Schrödinger picture.

a) Show that any $\mid \Psi(t)\rangle$ which solves (0.37) can be expressed as follows:

$$\mid \Psi(t)\rangle = e^{-\frac{i}{\hbar}Ht} \mid \Psi(0)\rangle \tag{0.40}$$

b) At time $t = 0$ the two states are related by

$$\mid \psi(0)\rangle = F \mid \phi(0)\rangle \tag{0.41}$$

Where F is in the Schr̈dingier picture and does not have any explicit time dependence. What is the condition on the operator F that this relation holds also for later times, i.e.

$$\mid \psi(t)\rangle = F \mid \phi(t)\rangle \tag{0.42}$$

c) If you transform the operator F from the Schrödinger picture to the Heisenberg picture, will it be time dependent ? Give an argument for your yes/no/depends answer.

Solution:

a) Since H is not time dependent it commutes with itself at all times. Therefore

$$\begin{aligned} i\hbar\frac{\partial}{\partial t} \mid \Psi(t)\rangle &= i\hbar\frac{\partial}{\partial t}e^{-\frac{i}{\hbar}Ht} \mid \Psi(0)\rangle \\ &= He^{-\frac{i}{\hbar}Ht} \mid \Psi(0)\rangle \\ &= H \mid \Psi(t)\rangle \end{aligned} \tag{0.43}$$

b) We start by multiplying

$$\mid \psi(0)\rangle = F \mid \phi(0)\rangle \tag{0.44}$$

from the left by $e^{-\frac{i}{\hbar}Ht}$ and inserting 1 in between F and $\mid \phi(0)\rangle$.

$$e^{-\frac{i}{\hbar}Ht} \mid \psi(0)\rangle = e^{-\frac{i}{\hbar}Ht}Fe^{+\frac{i}{\hbar}Ht}e^{-\frac{i}{\hbar}Ht} \mid \phi(0)\rangle \tag{0.45}$$

becomes

$$| \psi(t)\rangle = F(t) \mid \phi(t)\rangle \tag{0.46}$$

with

$$| F(t)\rangle = e^{-\frac{i}{\hbar}Ht} F e^{+\frac{i}{\hbar}Ht} \tag{0.47}$$

This expression is equal to F if

$$[H, F] = 0 \tag{0.48}$$

i.e. the operator F commutes with H and hence is a conserved quantity.

c) The operator F_H in the Heisenberg picture is defined to be

$$| F_H\rangle = e^{+\frac{i}{\hbar}Ht} F e^{-\frac{i}{\hbar}Ht} \tag{0.49}$$

Hence if $[F, H] = 0$ the operator $F_H = F$ and is time independent.

QUESTION 5: [20 points]

Consider the one dimensional periodic Ising chain with 3 sites

$$H = -\frac{j}{\hbar^2}\Big(S_1^z S_2^z + S_2^z S_3^z + S_3^z S_1^z\Big) + \frac{b}{\hbar}\Big(S_1^z + S_2^z + S_3^z\Big) \tag{0.50}$$

Where $\vec{S}_i$ is the spin 1/2 operator of the i-th site (treat the spins as distinguishable).

a) Find a normalized basis of eigenstates of H.

b) Find the spectrum of H and its degeneracy

c) For $b \neq 0$ calculate the partition function

$$Z = \mathrm{tr}(e^{-\beta H}) \tag{0.51}$$

d) Calculate the thermal expectation value of $S_1^z + S_2^z + S_3^z$ for the case $b = 0$

$$< S_1^z + S_2^z + S_3^z >= \mathrm{tr}\Big((S_1^z + S_2^z + S_3^z)e^{-\beta H}\Big)\ |_{b=0} \tag{0.52}$$

Solution:

a) Use the basis of eigenstates of $S^z = \frac{\hbar}{2}\sigma^3$ with

$$S^z \mid s = \pm 1\rangle = \pm\frac{\hbar}{2} \mid s = \pm 1\rangle \tag{0.53}$$

Since $[S_i^z, S_j^z] = 0$ for $i, j = 1, 2, 3$ the tensor product of the $\mid \pm\rangle$ are eigen states of H

$$\mid s_1, s_2, s_3\rangle = \mid s_1\rangle \otimes \mid s_2\rangle \otimes \mid s_3\rangle, \qquad s_{1,2,3} = \pm 1 \tag{0.54}$$

b) The eigenvalues of H are

$$E_{\{s\}} = -j\frac{1}{4}(s_1 s_2 + s_2 s_3 + s_3 s_1) + b\frac{1}{2}(s_1 + s_2 + s_3) \tag{0.55}$$

The values can easily be tabulated

$$\begin{array}{cccc}
\{s\} & s_1s_2 + s_2s_3 + s_3s_1 & (s_1 + s_2 + s_3) & E_{\{s\}} \\
+++ & 3 & 3 & -\frac{3j}{4} + \frac{3b}{2} \\
++- & -1 & 1 & \frac{j}{4} + \frac{b}{2} \\
+-+ & -1 & 1 & \frac{j}{4} + \frac{b}{2} \\
-++ & -1 & 1 & \frac{j}{4} + \frac{b}{2} \\
--+ & -1 & -1 & \frac{j}{4} - \frac{b}{2} \\
-+- & -1 & -1 & \frac{j}{4} - \frac{b}{2} \\
+-- & -1 & -1 & \frac{j}{4} - \frac{b}{2} \\
--- & +3 & -3 & -\frac{3j}{4} - \frac{3b}{2}
\end{array} \tag{0.56}$$

Hence there are four possible eigenvalues $-\frac{3j}{4}+\frac{3b}{2}$ with degeneracy 1, $\frac{j}{4}+\frac{b}{2}$ with degeneracy 3, $\frac{j}{4}-\frac{b}{2}$ with degeneracy 3 and $-\frac{3j}{4}+\frac{3b}{2}$ with degeneracy 1.

c) The trace is performed as the sum over the normalized eigenstates

$$\begin{aligned} Z &= tre^{-\beta H} \\ &= \sum_{\{s\}} \langle s_1, s_2, s_3 \mid e^{-\beta H} \mid s_1, s_2, s_3 \rangle \\ &= e^{\beta\frac{3j}{4}-\beta\frac{3b}{2}} + 3e^{-\beta\frac{j}{4}-\beta\frac{b}{2}} + 3e^{-\beta\frac{j}{4}+\beta\frac{b}{2}} + e^{\beta\frac{3j}{4}+\beta\frac{3b}{2}} \end{aligned} \tag{0.57}$$

d) Note that a derivative of b of Z inserts the operator $S_1^z + S_2^z + S_3^z$ into the trace. Z is symmetric under $b \to -b$ and hence only even powers appear in Z. Therefore the expectation value vanishes. One can of course compute this directly.

Solution Final 2012

Q1

a) $H = \hbar\omega \begin{pmatrix} 1 & & \\ & 3 & \\ & & 3 \end{pmatrix}$

Eigenvectors and eigenvalues

$|u_1\rangle \qquad EV_1 = \hbar\omega$

$|u_2\rangle \qquad EV_1 = 3\hbar\omega$

$|u_3\rangle \qquad EV_1 - 3\hbar\omega$

$|\psi(t=0)\rangle = c_1|u_1\rangle + c_2|u_2\rangle + c_3|u_3\rangle$

prob to obtain EV_i : $|c_i|^2$

measurement: $\boxed{\begin{array}{ll} \hbar\omega & \text{prop} = |c_1|^2 = \frac{1}{2} \\ 3\hbar\omega & \text{prob } |c_2|^2 + |c_3|^2 = \frac{1}{4} + \frac{1}{4} = 1/2 \end{array}}$

$$\boxed{\langle H \rangle = \frac{1}{2}\hbar\omega + \frac{1}{2}3\hbar\omega = 2\hbar\omega}$$

$$\boxed{\begin{array}{llll} A = +a & \text{prob} & \frac{1}{2}+\frac{1}{2} = 1 \\ A: -a & \text{prob} & 0 \end{array}}$$

After measurement of A is unchanged.

$$|\psi\rangle = |\psi(t=0)\rangle$$

c) $|\psi(t)\rangle = e^{-i/\hbar H t} |\psi(t=0)\rangle$

Since $|\psi\rangle$ is a linear combination of eigenvectors one has

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}} e^{-i/\hbar\, \hbar\omega t} |u_1\rangle + \frac{1}{2} e^{-i/\hbar\, 3\hbar\omega t} |u_2\rangle + \frac{1}{2} e^{-i/\hbar\, 3\hbar\omega t} |u_3\rangle$$

$$\boxed{|\psi(t)\rangle = \frac{1}{\sqrt{2}} e^{-i\omega t} |u_1\rangle + \frac{1}{2} e^{-i3\omega t} |u_2\rangle + \frac{1}{2} e^{-3i\omega t} |u_3\rangle}$$

$$\langle H^2\rangle - \langle H\rangle^2 = \frac{1}{2}(\hbar\omega)^2 + \frac{1}{2}(3\hbar\omega)^2 - (2\hbar\omega)^2$$

$$= 5(\hbar\omega)^2 - 4(\hbar\omega)^2$$

$$\boxed{\langle H^2\rangle - \langle H\rangle^2 = (\hbar\omega)^2}$$

b) Possible measurement results are given by eigenvalues

$$\det(A - \lambda \mathbb{1}) = \det\begin{pmatrix} a-\lambda & 0 & 0 \\ 0 & 0 & \lambda & a \\ 0 & 0 & a & \lambda \end{pmatrix}$$

$$= -(\lambda - a)^2(\lambda + a)$$

$\lambda = a$ (2 dim degenerate)

$\lambda = -a$

Eigenvectors $\lambda = a$ $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = |u_1\rangle$ & $\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}|u_2\rangle + \frac{1}{\sqrt{2}}|u_3\rangle$

$\lambda = -a$ $\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}|u_2\rangle - \frac{1}{\sqrt{2}}|u_3\rangle$

$$|\psi(t=0)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

c) continued..

Since $[A, H] = 0$ they have a common basis of eigenstates & $\langle A \rangle$ is not time dependent.

$$\langle A \rangle = +a$$

On the other hand since $[B, H] \neq 0$ the expectation value of B will depend on time:

$$\langle B \rangle = \left(\frac{1}{\sqrt{2}} e^{+i\omega t}, \frac{1}{2} e^{+3i\omega t}, \frac{1}{2} e^{3i\omega t} \right) b \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} e^{-i\omega t} \\ \frac{1}{2} e^{-3i\omega t} \\ \frac{1}{2} e^{-3i\omega t} \end{pmatrix}$$

$$= \left(\frac{1}{\sqrt{2}} e^{i\omega t}, \frac{1}{2} e^{3i\omega t}, \frac{1}{2} e^{3i\omega t} \right) \times \begin{pmatrix} \frac{1}{2} e^{-3i\omega t} \\ \frac{1}{\sqrt{2}} e^{-i\omega t} \\ \frac{1}{2} e^{-3i\omega t} \end{pmatrix}$$

$$\langle B \rangle = \frac{1}{2\sqrt{2}} e^{-2i\omega t} + \frac{1}{2\sqrt{2}} e^{+2i\omega t} + \frac{1}{4}$$

d) Since ~~A~~ $|\psi(t)\rangle$ only differs by phases from $|\psi(0)\rangle$ the measurement results & probabilities for A are unchanged

A: a with probability 1

$-a$ with " 0

Expand $|\psi(t)\rangle$ in terms of normalized eigenvectors of B:

$$|v_3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad |v_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad |v_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

EV = b $\qquad b \qquad -b$

$$|\psi(t)\rangle = \frac{1}{2} e^{-3i\omega t} |v_3\rangle + \frac{1}{2} e^{-3i\omega t} \frac{1}{\sqrt{2}}\left(|v_1\rangle + |v_2\rangle\right) + \frac{1}{\sqrt{2}} e^{-i\omega t} \frac{1}{\sqrt{2}}\left(|v_1\rangle - |v_2\rangle\right)$$

$$|\psi(t)\rangle = \frac{1}{2} e^{-3i\omega t} |v_3\rangle$$

$$+ \left(\frac{1}{2\sqrt{2}} e^{-3i\omega t} + \frac{1}{2} e^{-i\omega t} \right) |v_1\rangle$$

$$+ \left(\frac{1}{2\sqrt{2}} e^{-3i\omega t} - \frac{1}{2} e^{-i\omega t} \right) |v_2\rangle$$

Value:

b prop: $\frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{4\sqrt{2}} \left(e^{-2i\omega t} + e^{2i\omega t} \right)$

$-b$ prob: $\frac{1}{4} + \frac{1}{8} - \frac{1}{4\sqrt{2}} \left(e^{-2i\omega t} + e^{2i\omega t} \right)$

Check Σ prob $= 1$ ✓

Q2

$$H = J\vec{S}_1 \cdot \vec{S}_2$$
$$= J\left\{\frac{1}{2}(\vec{S}_1+\vec{S}_2)^2 - \frac{1}{2}\vec{S}_1^2 - \frac{1}{2}\vec{S}_2^2\right\}$$

a) we know from angular momentum:

$$[\vec{J}^2, J_i] = 0 \qquad (*)$$

4 commuting operators:

$$(\vec{S}_1+\vec{S}_2)^2 \qquad \vec{S}_1^2 \qquad \vec{S}_2^2 \qquad (S_1+S_2)_z$$

In tensor product operator 1 & 2 commute (**)

$[(\vec{S}_1+\vec{S}_2)^2, (S_1+S_2)_z] = 0$ because of (*)

$[(\vec{S}_1+\vec{S}_2)^2, \vec{S}_{1/2}^2] = [\vec{S}_{1/2}^2 + 2\vec{S}_1\vec{S}_2, \vec{S}_{1/2}^2] = 0$ because of (*) & (**)

$[\vec{S}_1^2, (\vec{S}_1+\vec{S}_2)_z] = 0$ because of (*) & (**)

b) eigenstates of $|j, j_{(1)}, j_{(2)}, j_z\rangle$

$$H|j, j_{(1)}, j_{(2)}, j_z\rangle$$

$$= \frac{J}{2}\hbar^2 \left(j(j+1) - j_1(j_1+1) - j_2(j_2+1) \right)$$

Range of $j = j_1 + j_2,\ j_1 + j_2 - 1, \ldots\ |j_1 - j_2|$

without loss of generality assume $j_1 \geq j_2$

- Largest eigenvalue j is largest $j = j_1 + j_2$.

$$E = \frac{J}{2}\hbar^2 \left\{ (j_1 + j_2)(j_1 + j_2 + 1) - j_1(j_1+1) - j_2(j_2+1) \right\}$$

$$= \frac{J}{2}\hbar^2 \left\{ j_1^2 + 2j_1j_2 + j_2^2 \cancel{+j_1} \cancel{+j_2} - \cancel{j_1^2} \cancel{-j_1} \cancel{-j_2} - j_1 \right\}$$

$$= \frac{J}{2}\hbar^2\, 2j_1j_2$$

$$= J\hbar^2 j_1 j_2$$

degeneracy: $2(j_1 + j_2) + 1$

Lowest eigenvalue j is smallest $j = j_1 - j_2$

$$E = \frac{J}{2}\hbar^2 \left\{ (j_1 - j_2)(j_1 - j_2 + 1) - j_1(j_1+1) - j_2(j_2+1) \right\}$$

$$= \frac{J}{2}\hbar^2 \left\{ \cancel{j_1^2} - 2j_1 j_2 + \cancel{j_2^2} + \cancel{j_1} - j_2 \right.$$
$$\left. - \cancel{j_1^2} - \cancel{j_1} - \cancel{j_2^2} - j_2 \right\}$$

$$= \frac{J}{2}\hbar^2 \left(-2 j_1 j_2 - 2 j_2 \right)$$

$$= - J\hbar^2 \, j_2 (j_1 + 1)$$

<u>degeneracy</u>: $2(j_1 - j_2) + 1$

~~c) $j_1 = 5/2$ $j_2 = 2$~~

~~values of $j = 7/2$ $5/2$ $3/2$ $1/2$~~

c) $j_1 = 5/2 \quad j_2 = 1$

Possible values of j

- $j = 7/2$ degeneracy : 8

$$E = \frac{J}{2}\hbar^2 \left(\frac{7}{2}\frac{9}{2} - 5/2\ 7/2 - 2\right)$$

$$= \frac{J}{8}\hbar^2 \quad (63 - 35 - 8)$$

$$= \frac{J}{8}\hbar^2 \ 20$$

$$= J\hbar^2 \frac{5}{2}$$

- $j = 5/2$ degeneracy 6

$$E = \frac{J}{2}\hbar^2 \left(\frac{5}{2}\frac{7}{2} - 5/2\ 7/2 - 2\right)$$

$$= -J\hbar^2$$

- $j = 3/2$ degeneracy 4

$$E = \frac{J}{2}\hbar^2 \left(\frac{3}{2} * 5/2 - 5/2\ 7/2 - 2\right)$$

$$= \frac{J}{8}\hbar^2 \quad 15 - 35 - 8 \quad = J\hbar^2 \frac{7}{?}$$

QUESTION 3: [20 points]

Consider two (distinguishable) spin 1/2 particles with Hamiltonian

$$H = h_0\Big(\sigma_x^{(1)} \otimes \sigma_x^{(2)} + \sigma_y^{(1)} \otimes \sigma_y^{(2)}\Big) \tag{0.8}$$

Where σ_i are the Pauli-matrices.

a) Show that

$$[\sigma_x^{(1)} \otimes \sigma_x^{(2)}, \sigma_y^{(1)} \otimes \sigma_y^{(2)}] = 0 \tag{0.9}$$

b) Calculate the spectrum of H

c) Calculate the state at time $t \mid \psi(t)\rangle$ with the initial condition

$$\mid \psi(t=0)\rangle = \mid j = \frac{1}{2}, m = \frac{1}{2}\rangle \otimes \mid j = \frac{1}{2}, m = \frac{1}{2}\rangle \tag{0.10}$$

Where m denotes the eigenvalue of σ_z.

d) For the state obtained in c) Calculate

$$\langle \psi(t) \mid \sigma_z \otimes \sigma_z \mid \psi(t)\rangle \tag{0.11}$$

Solution:

a) Using the distribution identity for the commutator

$$\begin{aligned}
[\sigma_x^{(1)}\sigma_x^{(2)}, \sigma_y^{(1)}\sigma_y^{(2)}] &= \sigma_x^{(1)}[\sigma_x^{(2)}, \sigma_y^{(1)}\sigma_y^{(2)}] + [\sigma_x^{(1)}, \sigma_y^{(1)}\sigma_y^{(2)}]\sigma_x^{(2)} \\
&= \sigma_x^{(1)}\sigma_y^{(1)}[\sigma_x^{(2)}, \sigma_y^{(2)}] + [\sigma_x^{(1)}, \sigma_y^{(1)}]\sigma_y^{(2)}\sigma_x^{(2)} \\
&= 2i\sigma_x^{(1)}\sigma_y^{(1)}\sigma_z^{(2)} + 2i\sigma_z^{(1)}\sigma_y^{(2)}\sigma_x^{(2)} \\
&= -2\sigma_z^{(1)}\sigma_z^{(2)} + 2\sigma_z^{(1)}\sigma_z^{(2)} \\
&= 0
\end{aligned} \tag{0.12}$$

Where we used the following property of the Pauli matrices.

$$\sigma_x\sigma_y = i\sigma_z, \quad \sigma_y\sigma_x = -i\sigma_z, \tag{0.13}$$

b) For eigenstates $\mid \pm\rangle$ of σ_z one has

$$\sigma_x \mid \pm\rangle = \mid \mp\rangle, \quad \sigma_y \mid \pm\rangle = \pm i \mid \pm\rangle \tag{0.14}$$

For the tensor product of two spins one has then

$$\begin{aligned}
\sigma_x^{(1)}\sigma_x^{(2)} \mid \pm\pm\rangle &= \mid \mp\mp\rangle \\
\sigma_x^{(1)}\sigma_x^{(2)} \mid \pm\mp\rangle &= \mid \mp\pm\rangle \\
\sigma_y^{(1)}\sigma_y^{(2)} \mid \pm\pm\rangle &= - \mid \mp\mp\rangle \\
\sigma_y^{(1)}\sigma_y^{(2)} \mid \pm\mp\rangle &= \mid \mp\pm\rangle
\end{aligned} \tag{0.15}$$

Hence the following linear combinations are eigenvectors of H with eigenvalues

$$\begin{aligned}
&\mid ++\rangle, \quad E = 0 \\
&\mid --\rangle, \quad E = 0 \\
&\frac{1}{\sqrt{2}}(\mid +-\rangle + \mid -+\rangle), \quad E = 2h_0 \\
&\frac{1}{\sqrt{2}}(\mid +-\rangle - \mid -+\rangle), \quad E = -2h_0
\end{aligned} \tag{0.16}$$

c) Using the result of a) one can calculate this directly

$$\begin{aligned}
\mid \psi(t)\rangle &= \exp\Big(-\frac{i}{\hbar}h_0(\sigma_x^{(1)}\sigma_x^{(2)} + \sigma_y^{(1)}\sigma_y^{(2)})t\Big) \mid ++\rangle \\
&= \exp\Big(-\frac{i}{\hbar}h_0\sigma_x^{(1)}\sigma_x^{(2)}t\Big)\exp\Big(-\frac{i}{\hbar}h_0\sigma_y^{(1)}\sigma_y^{(2)}t\Big) \mid ++\rangle \\
&= \exp\Big(-\frac{i}{\hbar}h_0\sigma_x^{(1)}\sigma_x^{(2)}t\Big)\Big(\cos h_0 t \mid ++\rangle + i\sin h_0 t \mid --\rangle\Big) \\
&= (\cos^2 h_0 t + \sin^2 h_0 t) \mid ++\rangle + (i - i)\sin h_0 t \cos h_0 t \mid --\rangle \\
&= \mid ++\rangle
\end{aligned} \tag{0.17}$$

Or one can also argue that $\mid ++\rangle$ is an eigenstate of H with eigenvalue 0 and that the time evolution is therefore trivial.

d) Since $\mid \psi(t)\rangle = \mid ++\rangle$ for all times (see c) one has

$$\begin{aligned}
\langle\psi(t) \mid \sigma_z^{(1)}\sigma_z^{(2)} \mid \psi(t)\rangle &= \langle ++ \mid \sigma_z^{(1)}\sigma_z^{(2)} \mid ++\rangle \\
&= +1\langle ++ \mid\mid ++\rangle \\
&= 1
\end{aligned} \tag{0.18}$$

Q4

a) $\langle n|H|n+1\rangle^* = \langle n+1|H|n\rangle$ since H is hermitian

$= \langle n'|H|n'-1\rangle \qquad n' = n+1$

$\Rightarrow \boxed{\langle n+1|H|n\rangle = r_1 e^{-ia_1}}$

$\langle n|H|n+2\rangle^* = \langle n+2|H|n\rangle$

$\Rightarrow \boxed{\langle n+2|H|n\rangle = r_2 e^{-ia_2}}$

b) $H = A_0 \mathbb{1} + A_1 T^+ + A_1^* T$

$+ A_2 T^{+2} + A_2^* T^2$

check:

$\langle n|H|n+1\rangle = A_0\langle n|n+1\rangle + A_1\langle n|T^+|n+1\rangle$

$+ A_1^*\langle n|T|n+1\rangle$

$+ A_2\langle n|T^{+2}|n+1\rangle$

$+ A_2^*\langle n|T^2|n+1\rangle$

$$= A_1 \langle n | n \rangle = A_1$$

<u>*Likewise:*</u>

$$\langle n | H | n+2 \rangle = A_0 \langle n | n+2 \rangle + A_1 \langle n | T^{+} | n+2 \rangle$$

$$+ A_1^* \langle n | T | n+2 \rangle$$

$$+ A_2 \langle n | T^{+2} | n+2 \rangle$$

$$+ A_2^* \langle n | T^2 | n+2 \rangle$$

$$= A_2 \langle n | n \rangle$$

$$= A_2$$

$$\langle n | H | n \rangle = A_0 \langle n | n \rangle$$

All other terms vanish since $\langle n | n+2 \rangle = 0$ & $\langle n | n+1 \rangle = 0$

c) $H = A_0 + A_1 T^+ + A_1^* T + A_2 T^{+2} + A_2^* T^2$

Bloch states:

$$|\alpha_m\rangle = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} e^{i\alpha_m n} |n\rangle \qquad \alpha_m = \frac{2\pi m}{N}$$

$$H|\alpha_m\rangle = A_0 |a_m\rangle + A_1 e^{ia_m} |a\rangle + A_1^* e^{-i\alpha_m} |a_m\rangle$$
$$+ A_2 e^{2ia_m} |a_m\rangle + A_2^* e^{-2ia_m} |a_m\rangle$$

since: $T|a_m\rangle = \frac{1}{\sqrt{N}} \sum e^{ia_m n} |n+1\rangle$

$$= e^{-i\alpha_m} |a\rangle$$

$$T^2 |a_m\rangle = e^{-2ia_m} |a\rangle$$

$$T^+ |a_m\rangle = e^{ia_m} |a\rangle$$

$$(T^+)^2 |a_m\rangle = e^{2ia_m} |a_m\rangle$$

$$H |a_m\rangle = E_m |a_m\rangle$$

$$E_m = A_0 + r_1 e^{i(\varphi_1 + a_m)} + r_1 e^{-i(\varphi_1 + a_m)}$$
$$+ r_2 e^{i(\varphi_2 + 2a_m)} + r_2 e^{-i(\varphi_2 + 2a_m)}$$

$$\boxed{E_m = A_0 + 2r_1 \cos\left(\varphi_1 + \frac{2\pi m}{N}\right) + 2r_2 \cos\left(\varphi_2 + \frac{4\pi m}{N}\right)}$$

d) $X|n\rangle = a_n |n\rangle$

- Matrix elements

$$\langle m|X|n\rangle = a_n \langle m|n\rangle -$$

$$= a_n \, \delta_{mn} = X_{mn}$$

$$\langle m|X|n\rangle^* = \langle n|X^+|m\rangle$$

$$= a_m \delta_{mn} = X^+_{nm}$$

Matrix elements $X^+_{nm} = X_{nm}$ hermitian

- $$T^+ X T \, |n\rangle = T^+ X |n+1\rangle$$

$$= a(n+1) T^+ |n+1\rangle$$

$$= a(n+1) \, |n\rangle$$

$$= a_n|n\rangle + a|n\rangle$$

$$= X|n\rangle + a|n\rangle$$

$$= (X+a)|n\rangle$$

Valid for all $|n\rangle$ hence the operator identity

$$\boxed{T^{+} X T = X + a}$$ holds

$\boxed{Q5}$

a) $\Pi_i' \psi' = \left(-i\hbar \frac{\partial}{\partial x^i} - e(A_i + \partial_i \Lambda)\right) e^{ie/\hbar \Lambda} \psi$

$= e^{ie/\hbar \Lambda}\left(\cancel{e \frac{\partial}{\partial x^i} \Lambda} - \cancel{e \frac{\partial}{\partial x_i} \Lambda} - i\hbar \frac{\partial}{\partial x^i} - eA_i\right)\psi$

$= e^{ie/\hbar \Lambda} \ \Pi_i \psi_i$

b) $[\Pi_1, \Pi_2] \cdot$ acting on wave function $\bullet$ to the right

$= \left(-i\hbar \frac{\partial}{\partial x_1} - eA_1\right)\left(-i\hbar \frac{\partial}{\partial x^2} - eA_2\right)$

$\left(-i\hbar \frac{\partial}{\partial x_2} - eA_2\right)\left(-i\hbar \frac{\partial}{\partial x^1} - eA_1\right) \cdot$

$= -\hbar^2 \underbrace{\cancel{\left(\frac{\partial}{\partial x^1}\frac{\partial}{\partial x^2} - \frac{\partial}{\partial x^2}\frac{\partial}{\partial x^1}\right)}}_{0} + e^2 \underbrace{\cancel{(A_1 A_2 - A_2 A_1)}}_{0}$

$+ ie\hbar\left(\frac{\partial}{\partial x^1} A_2\right) + \cancel{ei\hbar A_1 \frac{\partial}{\partial x^2}} + \cancel{ie\hbar A_2 \frac{\partial}{\partial x^1}}$

$- ie\hbar\left(\frac{\partial}{\partial x^2} A_1\right) - \cancel{ie\hbar A_1 \frac{\partial}{\partial x^2}} - \cancel{ie\hbar A_2 \frac{\partial}{\partial x^1}}$

$= ie\hbar\left(\frac{\partial}{\partial x^1} A_2 - \frac{\partial}{\partial x^2} A_1\right) \bullet$

$$B_i = \varepsilon_{ijk} \partial_j A_k$$

ie.

$$B_3 = \partial_1 A_2 - \partial_2 A_1$$

hence:

$$[\Pi_1, \Pi_2] \cdot = ie\hbar B_3 \cdot$$

since $\vec{B}$ is gauge invariant, the result is gauge invariant.

c) $$j_i = \frac{\hbar}{2mi}\left(\psi^* \frac{\partial}{\partial x^i}\psi - \psi \frac{\partial}{\partial x^i}\psi^* - \frac{2ie}{\hbar} A_i \psi^* \psi\right)$$

$$j_i' = \frac{\hbar}{2mi}\left(\cdot \psi^{*\prime} \frac{\partial}{\partial x^i}\psi' - \psi' \frac{\partial}{\partial x^i}\psi^{*\prime} - \frac{2ie}{\hbar} A_i' \psi^{*\prime}\psi\right)$$

$$= \frac{\hbar}{2m_i}\left(\psi^* e^{-ie/\hbar \Lambda} \frac{\partial}{\partial x^i}\left(e^{ie/\hbar \Lambda}\psi\right) - e^{ie/\hbar \Lambda}\psi \frac{\partial}{\partial x}\left(e^{-ie/\hbar \Lambda}\psi^*\right)\right.$$

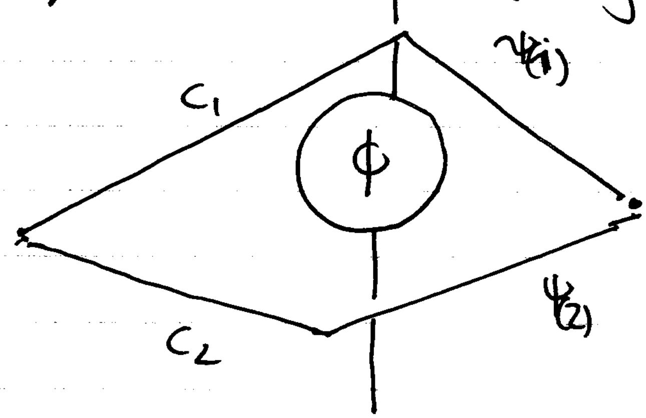
$$\left. - \frac{2ie}{\hbar}\left(A_i + \partial_i \Lambda\right) e^{-ie/\hbar \Lambda}\psi^* e^{ie/\hbar \Lambda}\psi\right)$$

$$j_i' = \frac{\hbar}{2m_i}\left\{\left(\frac{ie}{\hbar}\partial_i\Lambda \cancel{+ \frac{ie}{\hbar}\partial_i\Lambda}\right)\psi\psi^* - \frac{2ie}{\hbar}\cancel{\partial_i\Lambda}\,\psi^*\psi\right\} + j_i$$

$$= j_i$$

$j_i' = j_i$ ie probability current is gauge invariant

d) Consider a scattering experiment with a region



where $\psi = 0$ (infinite potential walls)

but nonzero Φ_B (magnetic flux)

$\psi_1^{(0)}(r)$

$\psi_2^{(0)}(r)$ wave functions without A

even though $\vec{B}=0$ outside solenoid A is nonzero.

$$\psi^{\Phi}_{(1)}(r) = e^{ie/\hbar \int_{C_1} d\vec{s}\,\vec{A}}\ \psi^0_{(1)}(r)$$

$$\psi^{\Phi}_{(2)}(r) = e^{ie/\hbar \int_{C_2} d\vec{s}\,\vec{A}}\ \psi^0_{(2)}(r)$$

total amplitude $\psi^0_{tot} = \psi^0_{(1)} + \psi^0_{(2)}$ without Φ

$$\psi^{\Phi}_{tot} = \psi^{\Phi}_{(1)} + \psi^{\Phi}_{(2)} = e^{ie/\hbar \int_{C_2} d\vec{s}\,\vec{A}} \times \left\{ e^{ie/\hbar \left(\int_{C_1} d\vec{s}\,\vec{A} - \int_{C_2} d\vec{s}\,\vec{A}\right)} \psi^0_{(1)} + \psi^0_{(2)} \right\}$$

relative phase $e^{ie/\hbar \left(\int_{C_1} - \int_{C_2}\right) d\vec{s}\,\vec{A}} = e^{ie/\hbar \oint_C ds\,A} = e^{ie/\hbar\, \Phi}$

$$= e^{ie/\hbar\,(B\cdot \pi r^2)}$$

measurable interference effect if magnetic field is changed

$e/\hbar\,\Phi = 2\pi n$ constructive $\quad \frac{e}{\hbar}\Phi = \pi(2n+1)$ destructive

PHYSICS 221B

Final Exam – Winter 2013

Tuesday March 19th, 2013, at 3pm - 6pm

- Please write clearly
- Print your name on every page used, including this one;
- Make clear which question and which part you are answering on each page.
- No core-dumps please !
- No books, notes, computers, or calculators are allowed during the exam;
- Please turn off all electronic devices.
- Problems are **not** in order of difficulty.

Good Luck !!

question	possible points	achieved points
1.	20	
2.	15	
3.	15	
4.	20	
5.	15	
Total	85	

Some possibly useful formulas

1. Harmonic oscillator for a Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 = \hbar\omega(a^\dagger a + \frac{1}{2}) \tag{0.1}$$

with

$$a = \frac{1}{\sqrt{2m\hbar\omega}}(m\omega x + ip) \qquad\qquad [a, a^\dagger] = 1 \tag{0.2}$$

2. The angular momentum algebra is $[J^1, J^2] = i\hbar J^3$, and two cyclic permutations thereof. The corresponding ladder operators are defined to be $J^\pm = J^1 \pm iJ^2$, and act by

$$J^\pm |j, m\rangle = \hbar\sqrt{j(j+1) - m(m \pm 1)}\, |j, m \pm 1\rangle \tag{0.3}$$

3. spherical Bessel functions j_l

$$j_0(x) = \frac{\sin x}{x} \tag{0.4}$$

$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x} \tag{0.5}$$

4. Spherial harmonics

$$Y_0^0 = \frac{1}{\sqrt{4\pi}} \tag{0.6}$$

$$Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos\theta \tag{0.7}$$

$$Y_1^{\pm 1} = \mp\sqrt{\frac{3}{8\pi}} \sin\theta e^{\pm i\phi} \tag{0.8}$$

5. Scattering formula

$$\begin{aligned} e^{i\mathbf{k}\cdot\mathbf{r}} &= \sum_{\ell=0}^{\infty} i^\ell (2\ell+1) j_\ell(kr) P_\ell(\cos\theta) \\ \psi_{\mathbf{k}}(\mathbf{r}) &= \frac{1}{(2\pi)^{3/2}}\left[e^{i\mathbf{k}\cdot\mathbf{r}} + f(\mathbf{k}', \mathbf{k})\frac{e^{ikr}}{r}\right] \\ f(\mathbf{k}', \mathbf{k}) &= \sum_{\ell=0}^{\infty} (2\ell+1)\frac{e^{2i\delta_\ell} - 1}{2ik} P_\ell(\cos\theta) \end{aligned} \tag{0.9}$$

6. phase shift for spherical symmetric potential, 1st Born approximation

$$\delta_l = -k \int_0^\infty dr r^2 U(r) (j_l(kr))^2 \tag{0.10}$$

7. Possible useful integrals:

$$\int_0^{2\pi} d\phi \, e^{ia \sin\phi} = 2\pi J_0(|a|) \tag{0.11}$$

$$\int_0^{\pi} d\theta \sin\theta J_0(b \sin\theta) = \frac{2 \sin b}{b} \tag{0.12}$$

where J_0 is the Bessel function of the first kind.

QUESTION 1: [20 points]

We consider the Hamiltonian of a rigid rotator given by

$$H_0 = \frac{\vec{L}^2}{2I}$$

Where I is the moment of inertia.

It is assumed that the rigid rotator has a magnetic moment and is placed in an uniform electric field in the z-direction. Averaging over the radial dependence this amounts to adding a perturbation

$$H' = -\epsilon\mu\cos\theta$$

(i.e. For the problem you can neglect any radial dependence and treat the problem as one which only depends on the angular coordinates θ and ϕ).

a) Find the spectrum and degeneracies of H_0.

b) Using the following relation

$$\cos\theta\, Y_l^m = \sqrt{\frac{(l+1)^2 - m^2}{(2l+1)(2l+3)}}Y_{l+1}^m + \sqrt{\frac{l^2 - m^2}{(2l+1)(2l-1)}}Y_{l-1}^m$$

where Y_l^m are the spherical harmonics in standard spherical coordinates. Calculate the following matrix elements

$$\langle lm|\cos\theta|l'm'\rangle$$

Hint: very few of the matrix elements are nonzero.

c) Using the results of b) argue that: First, the first order contribution in perturbation theory to the energy of the state $|\, lm\rangle$ vanishes. Second, even though the spectrum is degenerate one can apply second order perturbation theory for the energy of the state $|\, lm\rangle$.

d) Calculate the second oder contribution to the shift in the energy for the state $|\, lm\rangle$.

QUESTION 2: [15 points]

Consider the scattering of a spinless particle of mass m from a diatomic molecule. The incoming momentum is $\vec{p} = \hbar k \hat{e}_z$ Assume that the molecule is much heavier than the scattering particle and that there is no recoil. The two atoms in the molecule are aligned along the y-axis and localized at $y = b$ and $y = -b$. The potential the particle feels in the presence of the molecule can be modeled by delta functions:

$$V(\vec{x}) = \alpha\Big(\delta(y-b)\delta(x)\delta(z) + \delta(y+b)\delta(x)\delta(z)\Big)$$

a) Calculate the scattering amplitude in the first Born approximation.

b) Calculate the differential cross section from a) (Express the result in terms of the scattering angles).

c) Calculate the total cross section. You can **either** do the integrals exactly **or** calculate the total cross section to order k^2 (inclusive) in the small k limit.

QUESTION 3: [15 points]

A two state system is described by the following Hamiltonian

$$H = H_0 + V(t)$$

With a time independent H_0 and a two orthonormal basis vectors satisfying

$$H_0 \mid 1\rangle = \epsilon_1 \mid 1\rangle, \quad H_0 \mid 2\rangle = \epsilon_2 \mid 2\rangle$$

The perturbation satisfies

$$V(t) \mid 1\rangle = \hbar\omega_1 e^{-i\omega t} \mid 2\rangle, \quad V(t) \mid 2\rangle = \hbar\omega_1 e^{i\omega t} \mid 1\rangle$$

a) Find the eigenvalues and eigenvectors of H

b) Solve the time dependent Schrödinger equation for $t > 0$ for a state with initial condition

$$\mid \psi(t = 0)\rangle = \mid 1\rangle$$

c) Calculate the probability to find the system at time $t > 0$ in the state $\mid 2\rangle$.

QUESTION 4: [20 points]

A particle is scattered by a spherical symmetric potential at energies which are low enough so that only the phase shifts δ_0 and δ_1 are nonzero. (For part a)-c) treat δ_0, δ_1 as given).

a) Show that the differential cross section is of the form

$$\frac{d\sigma}{d\Omega} = A + B\cos\theta + C\cos^2\theta$$

b) Determine A, B, C in terms of the phase shifts

c) Calculate the total cross section in terms of A, B, C.

d) Consider a very weak and short range potential (which behaves not worse than $1/r$ at the origin). Estimate the k dependence of δ_0 and δ_1 in the limit $k \to 0$.

QUESTION 5: [15 points]

Consider the one dimensional harmonic oscillator with Hamiltonian

$$H_0 = \frac{p^2}{2m} + \frac{1}{2} m\, \omega^2 x^2$$

At time $t > 0$ the following perturbation is turned on

$$H'(t) = \alpha\, x\, e^{-\frac{t}{\tau}}$$

a) If at time $t < 0$ the system is in its ground state (of H_0) calculate to first order in time dependent perturbation theory the probability that the system is found at time $t > 0$ in the first excited state.

b) If at time $t < 0$ the system is in the first excited state (of H_0) calculate to first order in time dependent perturbation theory the probability that the system is found at time $t > 0$ in the ground state.

c) For the harmonic oscillator with Hamiltonian H_0 above, give an example a of an adiabatic change and a sudden change. What is the time scale which is used to decide whether an adiabatic or sudden change approximation is appropriate ?

If the system is in the ground state at time $t = 0$ describe (without calculation) how the state evolves at later times for the two cases.

PHYSICS 221B

Practice Final Exam – Winter 2013

Real Exam: Tuesday March 19th, 2013, at 3pm - 6pm

- Please write clearly
- The order of problems is not by difficulty.
- Print your name on every page used, including this one;
- Make clear which question and which part you are answering on each page.
- No core-dumps please !
- No books, notes, computers, or calculators are allowed during the exam;
- Please turn off all electronic devices.

Good Luck !!

question	possible points	achieved points
1.	15	
2.	15	
3.	10	
4.	15	
5.	20	
Total	75	

Some possibly useful formulas

- Electric and magnetic fields:

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}, \quad \vec{B} = \vec{\nabla} \times \vec{A} \tag{0.1}$$

- Harmonic oscillator for a Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 = \hbar\omega(a^\dagger a + \frac{1}{2}) \tag{0.2}$$

with

$$a = \frac{1}{\sqrt{2m\hbar\omega}}(m\omega x + ip) \qquad [a, a^\dagger] = 1 \tag{0.3}$$

- The angular momentum algebra is $[J^1, J^2] = i\hbar J^3$, and two cyclic permutations thereof. The corresponding ladder operators are defined to be $J^\pm = J^1 \pm iJ^2$, and act by

$$J^\pm|j, m\rangle = \hbar\sqrt{j(j+1) - m(m \pm 1)}\,|j, m \pm 1\rangle \tag{0.4}$$

- Radial wave function R_{nl} for Hydrogen like atom with $V(r) = -Ze^2/r$.

$$R_{10}(r) = \left(\frac{Z}{a_0}\right)^{3/2} 2e^{-\frac{Zr}{a_0}} \tag{0.5}$$

$$R_{20}(r) = \left(\frac{Z}{2a_0}\right)^{3/2} (2 - \frac{Zr}{a_0})e^{-\frac{Zr}{2a_0}} \tag{0.6}$$

$$R_{21}(r) = \left(\frac{Z}{2a_0}\right)^{3/2} \frac{Zr}{\sqrt{3}a_0} e^{-\frac{Zr}{2a_0}} \tag{0.7}$$

- Bohr radius

$$a_0 = \frac{\hbar^2}{m_e e^2} \tag{0.8}$$

- Spherial harmonics

$$Y_0^0 = \frac{1}{\sqrt{4\pi}} \tag{0.9}$$

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta \tag{0.10}$$

$$Y_1^{\pm 1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\phi} \tag{0.11}$$

- spherical Bessel functions j_l

$$j_0(x) = \frac{\sin x}{x} \tag{0.12}$$

$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x} \tag{0.13}$$

- Legendre polynomials P_l.

$$P_0(x) = 1, \quad P_1(x) = x \tag{0.14}$$

- The Laplacian in spherical coordinates

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{\vec{L}^2}{\hbar^2 r^2} \tag{0.15}$$

- Potentially useful integrals

$$\int_0^\infty dx\; x^n e^{-ax} = a^{-1-n}\Gamma(n+1) \tag{0.16}$$

$$\int_{R^3} d^3x \frac{e^{i\vec{x}\cdot\vec{y}}}{|\vec{x}|} = \frac{4\pi}{|\vec{y}|^2} \tag{0.17}$$

$$\int_{-\infty}^\infty dx \frac{1}{1+x^2} = \pi \tag{0.18}$$

QUESTION 1: [15+5 points]

A relativistic particle in one dimension with mass m is subject to a harmonic oscillator potential, and governed by the following Hamiltonian,

$$H_c = \sqrt{m^2c^4 + p^2c^2} - mc^2 + \frac{1}{2}m\omega^2 x^2 \tag{0.19}$$

where $[x, p] = i\hbar$.

a) Show that n the limit $c \to \infty$, the Hamiltonian H_c reduces to the standard non-relativistic harmonic oscillator Hamiltonian (which will be denoted here by H_∞).

b) Using perturbation theory in power of $1/c^2$, compute the leading relativistic correction to the ground state energy of H_∞.

c) For the general case of finite c show that, in a basis where p is **diagonal**, the spectrum of H_c may be obtained by solving a Schrödinger-like differential equation.

QUESTION 2: [15 points]

An electron scatters off a hydrogen *atom* in the ground state. Ignore the effects due to the spin and the indistinguishability of the two electrons. The potential seen by the scattered electron is then given by,

$$V(r) = -\frac{\alpha}{r} + \alpha \int d^3y \frac{\rho(\vec{y})}{|\vec{x} - \vec{y}|} \tag{0.20}$$

where $r = |\vec{x}|$, α is the fine structure constant, and $\rho(\vec{x})$ is the normalized probability density of the bound electron.

a) Calculate the scattering amplitude f in the Born approximation, as a function of the Fourier transform of $\rho(\vec{x})$.

b) Evaluate the differential cross section as a function of $q = k \sin(\theta/2)$ for

$$\rho(\vec{x}) = \frac{1}{\pi a^3} e^{-2r/a} \tag{0.21}$$

corresponding to the probability density in the ground state of the Hydrogen atom, where $a = \hbar/(mc\alpha)$ is the Bohr radius.

c) Show that the total cross section is finite in the limit $k \to 0$.

QUESTION 3: [10 points]

a) Verify that, outside the range of a short-range potential, the wave functions

$$\begin{aligned} u_s(r,\theta) &= \frac{1}{r}e^{ikr} \\ u_p(r,\theta) &= (\frac{1}{r}+\frac{i}{kr^2})e^{ikr}\cos\theta \end{aligned} \tag{0.22}$$

represent outgoing s- and p-waves respectively.

b) A beam of particles represented by a plane wave with

$$\phi_k(\vec{r}) = \frac{1}{(2\pi)^{\frac{3}{2}}}\sum_{l=0}^{\infty} i^l(2l+1)j_l(kr)P_l(\cos\theta) \tag{0.23}$$

is scattered by an impenetrable sphere of radius R, where $kR << 1$. By considering only s and p wave components in the scattered wave, show that, to order $(kR)^2$, the differential cross-section for scattering at an angle θ is of the form

$$\frac{d\sigma}{d\Omega} = A + B\cos\theta \tag{0.24}$$

and compute A and B to order $\mathcal{O}(k^2R^2)$ included.

QUESTION 4: [15 points]

A hydrogen atom is located in a homogenous electric field (and vanishing magnetic field).

$$\vec{E} = \begin{pmatrix} 0 \\ 0 \\ E_z(t) \end{pmatrix}, \qquad E(t) = \frac{B\tau}{\pi e} \frac{1}{t^2 + \tau^2} \tag{0.25}$$

Neglect spin and any fine structure corrections for the Hamiltonian of the hydrogen atom.

a) Determine the time dependent perturbation $V(t)$ coming from the time dependent electric field

b) At time $t = -\infty$ the hydrogen atom is in its ground state. Calculate, to first order in time dependent perturbation theory,the probability of finding the atom at the time $t = \infty$ in the $2p$ state.

c) At time $t = -\infty$ the hydrogen atom is in its ground state. Calculate, to first order in time dependent perturbation theory, the probability of finding the atom at the time $t = \infty$ in the $2s$ state.

QUESTION 5: [20 points]

Consider the Hamiltonian for a "spin top" with principal moments of inertia I_1, I_2, I_3 with Hamiltonian

$$H = \frac{1}{2}\left(\frac{J_1^2}{I_1} + \frac{J_2^2}{I_2} + \frac{J_3^2}{I_3}\right) \tag{0.26}$$

Where $\sum J_i^2$ has eigenvalues $\hbar^2 j(j+1)$ with $j = 0, \frac{1}{2}, 1, \cdots$.

a) For the special case of a symmetric top, $I = I_1 = I_2 \neq I_3$, derive all the energy levels, and their respective degeneracies.

b) A slightly asymmetric top has $I_1 = I + \Delta$ and $I_2 = I - \Delta$. and assume that $|\Delta| \ll I$, and $|\Delta| \ll |I - I_3|$. Compute the energy eigenvalue of the $j = 0$ state, up to first order in Δ in perturbation theory.

c) Consider the three states with $j = 1$, does one have to use degenerate or non degenerate perturbation theory to calculate the energy shift to first order in Δ in perturbation theory ? Give an argument

d) Calculate energy shifts of the $j = 1$ states to first order in Δ.

Solutions for practice final

1

a) Expand H_c

$$H_c = mc^2\sqrt{1+\frac{p^2}{m^2c^2}} - mc^2 + \frac{1}{2}m\omega^2x^2$$

$$= \cancel{mc^2} + \frac{1}{2m}p^2 - \frac{1}{8}\frac{p^4}{m^3c^2} - \cancel{mc^2}$$

$$+ \frac{1}{2}m\omega^2x^2 + O(p^6)$$

In the limit $c\to\infty$ we have

$$\lim_{c\to\infty} H_c = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2x^2$$

i.e the standard Harmonic oscillator

b) $$H_0 = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2x^2$$

$$H_1 = -\frac{1}{8}\frac{p^4}{m^3c^2}$$

Rewrite

$$H_0 = \hbar\omega(a^+a + 1/2)$$

$$a - a^+ = i\sqrt{\frac{2}{m\hbar\omega}}\, p$$

$$\Rightarrow \quad p = \frac{1}{i}\sqrt{\frac{m\hbar\omega}{2}}\,(a - a^+)$$

$$\Rightarrow \quad H_1 = -\frac{\hbar^2\omega^2}{32mc^2}\,(a - a^+)^4$$

Since spectrum of H_0 is non-degenerate we can use non-degerate perturbation theory.

1st order shift in ground state energy

$$\Delta_0 = \frac{\langle 0 | H_1 | 0 \rangle}{\langle 0 | 0 \rangle}$$

Need to calculate

$$\langle 0 | (a - a^+)^4 | 0 \rangle$$

Binomial:

only terms with 2 a 2 a^+ survive.

$$\langle 0|(a-a^+)^4|0\rangle$$

$$= \langle 0|\ a^2 a^{+2} + a a^+ a a^+\ |0\rangle$$

$$= 3 \langle 0|0\rangle$$

$$\Rightarrow \quad \Delta_0 = -\frac{3\hbar^2\omega^2}{32 m c^2}$$

c) In momentum basis

$$\hat{p}\,|p\rangle = p\,|p\rangle$$

$$x\,|p\rangle = i\hbar \frac{\partial}{\partial p}\,|p\rangle$$

H acting in momentum basis

$$H_c = \sqrt{m^2c^4 + p^2c^2} - mc^2 \quad \dot{-} \frac{1}{2} m(\omega\hbar)^2 \frac{\partial^2}{\partial p^2}$$

is of the form:

$$\left(C_1' \frac{\partial^2}{\partial p^2} + V(p) \right) |\psi\rangle = E |\psi\rangle$$

where: $C = -\frac{1}{2} m(\omega\hbar)^2$

$V(p) = \sqrt{m^2c^4 + p^2c^2} - mc^2$

i.e a Schrödinger like equation.

Problem 2)

a) $U = \frac{2m}{\hbar^2} V$

Born approximation

$$f^{(1)}(k,k') = -\frac{1}{4\pi} \int d^3y \, e^{-i(\vec{k}'-\vec{k})\vec{y}} \, U(y)$$

set $\vec{q} = \vec{k}' - \vec{k}$ & insert U

$$f^{(1)}(\vec{q}) = -\frac{m\alpha}{2\pi\hbar^2} \int d^3y \, e^{-i\vec{q}\vec{y}} \left(-\frac{1}{|y|} + \int d^3w \frac{\rho(\vec{w})}{|\vec{y}-\vec{w}|} \right)$$

Use integral $\int d^3x \, \frac{e^{i\vec{x}\vec{y}}}{|\vec{x}|} = \frac{4\pi}{|\vec{y}|^2}$

& shift integration $\vec{y}' = \vec{y} - \vec{w}$

$$f^{(1)}(\vec{q}) = \frac{2m\alpha}{\hbar^2} \frac{1}{|\vec{q}|^2} - \frac{m\alpha}{2\pi\hbar^2} \int d^3y' \frac{e^{-i\vec{q}\vec{y}'}}{|\vec{y}'|} \times \int d^3w \, \rho(w) \, e^{-i\vec{q}\vec{w}}$$

$$\boxed{f^1(\vec{q}) = \frac{2m\alpha}{\hbar^2} \frac{1}{|\vec{q}|^2} \left(1 - \tilde{\rho}(\vec{q}) \right)}$$

b) Calculate the Fourier transform of $\varsigma(x)$ with

$$\varsigma(x) = \frac{1}{\pi a^3} e^{-2r/a}$$

$$\tilde{\varsigma}(q) = \int d^3 w \ \frac{1}{\pi a^3} e^{-2|w|/a} \cdot e^{i\vec{q}\vec{w}}$$

$$= \frac{1}{\pi a^3} \int d\varphi \int_{+1}^{+1} d(\cos\theta) \int d|w| \, |w|^2$$

$$\times e^{iq|w|\cos\theta - \frac{2|w|}{a}}$$

$$= -\frac{2}{a^3} \int dw \frac{|w|}{iq} \quad e^{(-iq - \frac{2}{a})|w|} - e^{(+iq - \frac{2}{a})|w|}$$

$$= \frac{-2}{a^3 i q} \left(\frac{1}{(+iq + \frac{2}{a})^2} - \frac{1}{(-iq + \frac{2}{a})^2} \right)$$

$$\cdot \underbrace{\int dw \, w \, e^{-|w|}}_{1} \qquad \text{see } (0.13)$$

$$\tilde{\rho}(q) = \frac{16}{a^4} \frac{1}{\left(q^2 + \frac{4}{a^2}\right)^2}$$

$$\Rightarrow f^{(1)}(q) = \frac{2m\alpha}{\hbar^2} \frac{1}{|q|^2} \left(1 - \frac{16}{a^4} \frac{1}{\left(q^2 + \frac{4}{a^2}\right)^2}\right)$$

$$\boxed{\begin{aligned} \frac{d\sigma}{d\Omega} &= |f^{(1)}|^2 \\ &= \frac{4m^2\alpha^2}{\hbar^4} \frac{1}{|q|^4} \left(1 - \frac{16}{a^4} \frac{1}{\left(|q|^2 + \frac{4}{a^2}\right)^2}\right)^2 \end{aligned}}$$

$$|q| = 2k \sin\frac{\theta}{2}$$

c) in the limit $h \to 0$ we have.

$$1 - \frac{16}{a^4} \frac{1}{\left(|q|^2 + \frac{4}{a^2}\right)^2} = 1 - \frac{1}{\left(\frac{a^2 q^2}{4} + 1\right)^2}$$

$$= 1 - 1 + \frac{a^2 q^2}{2} + o(q^2)$$

hence:

in the limit $k \to 0$ $\quad |q| \to 0$

$$\frac{d\sigma}{d\Omega} = \frac{4m^2\alpha^2}{k^4}\frac{1}{|q|^4}\left[\left(\frac{a^2 |q^2|}{2}\right)^2 + O(q^4)\right]$$

$$= \frac{\cancel{4}m^2\alpha^2}{k^2}\frac{a^4}{\cancel{4}} \qquad \text{constant.}$$

hence $\sigma_{tot} = \int \frac{d\sigma}{d\Omega} d\Omega$ is <u>finite</u>.

Problem 3

a) • the behavior $e^{ikr-i\omega t}$ of the wave functions means that the spherical wave propagates outward as an r with a constant phase grows as t increases.

[You can calculate the flux vector and it will point outward]

• u_s no angular dependence $\Rightarrow$ s-wave

u_p θ dependence $Y_1^0(\theta,\varphi)$

i.e. eigenstate of $\vec{L}^2$ with $l=1 \Rightarrow$ p-wave

• u_s & u_p have to be eigenstates of $\Delta = \vec{\nabla}^2$

Calculate:

$$\left(\frac{\partial}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}\right)u_s=\left(\frac{\partial}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}\right)\frac{1}{r}e^{ikr}$$

$$\left(+ \cancel{\frac{2}{r^3}} - \cancel{\frac{2ik}{r^2}} - \frac{k^2}{r} - \cancel{\frac{2}{r^3}} + \cancel{\frac{2ik}{r^2}} \right) e^{ikr}$$

$$= -k^2 u_s \quad \checkmark$$

$$\left(\frac{\partial}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} - \frac{2}{r^2} \right) u_p$$

$$= \left(\frac{\partial}{\partial r^2} + \frac{2}{r} \frac{\partial}{r} - \frac{2}{r^2} \right) \left(\frac{1}{r} e^{ikr} + \frac{i}{kr^2} e^{ikr} \right)$$

$$= -k^2 \frac{1}{r} e^{ikr} - \frac{2}{r^3} e^{ikr}$$

$$+ \left(\frac{i6}{kr^4} + \frac{4}{r^3} - \frac{ik}{r^2} - \frac{4i}{kr^4} - \frac{2}{r^3} - \frac{2i}{kr^4} \right) e^{ikr}$$

$$= -k^2 \left(\frac{1}{r} + \frac{i}{kr^2} \right) e^{ikr}$$

$$= -k^2 u_p \quad \checkmark$$

b) The total wave function is the superposition of incoming plane and outgoing spherical for $l=0$, $l=1$ spherical waves

we can use the basis function u_s, u_p for the later.

$$\psi_{tot} = \frac{1}{(2\pi)^{3/2}} \left\{ j_0(kr)\, P_0(\cos\theta) + i\, 3\, j_1(kr)\, P_1(\cos\theta) \right.$$

$$\left. + \frac{1}{(2\pi)^{3/2}} \left\{ c_0\, u_s(r,\theta) + c_1\, 3i\, u_p(r,\theta) \right\} \right.$$

+ higher spherical waves.

The hard sphere condition implies that

$$\psi_{tot}(r=R) = 0$$

plugging in the form of j_l & P_l and matching terms for $l=0$ & $l=1$ separately gives

$l=0$ $$\frac{\sin kR}{kR} + c_0 \frac{1}{R} e^{ikR} = 0$$

$l=1$ $$\left(\frac{\sin kR}{(kR)^2} - \frac{\cos kR}{kR}\right) + c_1 \left(\frac{1}{R} + \frac{i}{kR^2}\right) e^{ikR} = 0$$

Taylor expanding the sin, cos & e^{ikR}

$$C_0 = -R \left(1 - \frac{(kR)^2}{6}\right)\left(1 - ikR - \frac{(kR)^2}{2}\right) + O(kR)^3$$

$$= -R\left(1 - ikR - \frac{2}{3}(kR)^2\right)$$

$$C_1 = -R \frac{\left(\frac{1}{kR} - \frac{kR}{6} - \frac{1}{kR} + \frac{1}{2} kR + O(kR)^3\right)}{\left(1 + \frac{i}{kR}\right)} \times e^{-ikR}$$

$$C_1 = -R\left(+\frac{i}{3} k^2 R^2\right) + O(k^3 R^3)$$

in the limit $r \to \infty$

$$\psi_{tot} \sim \frac{1}{(2\pi)^{3/2}} \left(e^{i\vec{k}\vec{r}} + \frac{e^{ikr}}{r} f \right) + \ldots$$

We can read off f from leading $\frac{1}{r}$ behavior of outgoing spherical wave:

$$f = c_0 + 3ic_1 \cos\theta$$

$$\frac{d\sigma}{d\Omega} = |f|^2$$

since c_1 is of order k^2R^2 $|f|^2$ ~~to this order only contains~~ the term with $\cos^2\theta$ is of order k^4R^4 and hence dropped

$$\Rightarrow \boxed{\frac{d\sigma}{d\Omega} = A + B\cos\theta}$$

$$\frac{d\sigma}{d\Omega} = (c_0 + 3i\, c_1 \cos\theta)(\overline{c_0} - 3i\, \overline{c_1} \cos\theta)$$

$$= \underbrace{|c_0|^2}_{A} + \underbrace{3i(\overline{c_0}\, c_1 - c_0 \overline{c_1})}_{B} \cos\theta + \ldots$$

hence

$$A = |c_0|^2 .$$

$$= R^2 \left(1 - ikR - \frac{2}{3}(kR)^2\right)$$

$$\left(1 + ikR - \frac{2}{3}(kR)^2\right)$$

$$= R^2 \left(1 + (kR)^2 - \frac{4}{3}(k^2R^2) + \ldots\right)$$

$$\boxed{A = R^2\left(1 - \frac{1}{3}(kR)^2 + o(kR^4)\right)}$$

$$B = 3i\left(-R\ \frac{i}{3} k^2R^2 - (R)\left(-\frac{i}{3} k^2R^2\right)\right)$$

$$\boxed{B = 2k^2R^2 + o(k^2R^4)}$$

Problem 4.

a) Set $\vec{A} = 0$ & $E = -\nabla \Phi$

then

$$\phi = Z \cdot \frac{B\tau}{\pi e} \frac{1}{t^2 + T^2}$$

potential

$$\boxed{V(t) = e\phi = Z \cdot \frac{B\tau}{\pi} \frac{1}{t^2+T^2}}$$

b) $$C_n(t) = C_n(t_i) - \frac{i}{\hbar} \int_{t_1}^{t} dt' \sum_m \langle n | V_I(t') | m \rangle \, C_m(t')$$

1st order time dep perturbation theory.

$t_i = -\infty \qquad t = +\infty$

$n =$ 2p state

$m =$ 1s state 100 % at time $t = -\infty$

$$C_{2p}(t=+\infty) = -\frac{i}{\hbar} \int_{-\infty}^{+\infty} dt' \langle 2p | V | 1s \rangle$$

$$C_{2p}(t=+\infty) = -\frac{i}{\hbar} \int_{-\infty}^{+\infty} dt' \frac{B\tau}{\pi} \frac{1}{t^2+\tau^2} \cdot \langle 2p | z | 1s \rangle$$

- Integral $\displaystyle\int_{-\infty}^{+\infty} dx \frac{1}{1+x^2} = \pi$

$$\int dt' \frac{B\tau}{\pi} \frac{1}{t^2+\tau^2} = B$$

- Matrix element.

$$z = r\cos\theta$$

$$= \sqrt{\frac{4\pi}{3}}\, r\, Y_1^0$$

Wigner Eckhart only non vanishing matrix element $\langle \ell=1, m=0 | Z | \ell=0, m=0 \rangle$

$$\langle 2p | Z | 1s \rangle$$

$$= \int dr\, r^2 \int d\theta \sin\theta\, d\varphi\; R_{20}(r)\, Y_1^0 \cdot r \sqrt{\frac{4\pi}{3}}\, r\, Y_1^0 \cdot R_{10}\, Y_0^0$$

$$= \int d\theta \sin\theta\, d\varphi\; Y_1^0\, Y_1^0 \sqrt{\frac{1}{3}} \cdot \int dr\, r^3 R_{20} R_{10}$$

$$= \sqrt{\frac{1}{3}} \int dr\, r^3 R_{20} R_{10}$$

$$= \sqrt{\frac{1}{3}} \cdot \left(\frac{Z}{a_0}\right)^3 \frac{1}{\sqrt{2}} \int dr\, r^3 \left(2 - \frac{Zr}{a_0}\right) e^{-\frac{3}{2}\frac{Z}{a_0} r}$$

$$= \frac{1}{\sqrt{6}} \left\{ \left(\frac{2}{3}\right)^3 2 \int dx\, x^3 e^{-x} - \left(\frac{2}{3}\right)^4 \int dx\, x^4 e^{-x} \right\}$$

$$= \frac{1}{\sqrt{6}} \left\{ \frac{16}{27} \cdot 6 - \frac{16}{81} 4 \cdot 3 \cdot 2 \right\}$$

$$= \frac{1}{\sqrt{6}} \left(\frac{32}{9} - \frac{256}{27} \right)$$

$$= \frac{1}{\sqrt{6}} \left(\frac{96 - 256}{27} \right)$$

$$= \frac{1}{\sqrt{6}} \left(\frac{160}{27} \right)$$

$$\text{Prob} = |C_{2p}|^2 = \frac{B^2}{\hbar^2} \cdot \frac{1}{6} \left(\frac{160}{27} \right)^2$$

c) since $\langle 2s | z | 1s \rangle$ is zero by Wigner Eckhart the probability is zero

or directly by integral $\int d\varphi \, d\theta \cos\theta = 0$

Problem 5:

$$H = \frac{1}{2}\left(\frac{1}{I_1} J_1^2 + \frac{1}{I_2} J_2^2 + \frac{1}{I_3} J_3^2\right)$$

a) symmetric top

$$H_{sym} = \frac{1}{2I}\left(J_1^2 + J_2^2\right) + \frac{1}{2I_3} J_3^2$$

$$\vec{J}^2 = J_1^2 + J_2^2 + J_3^2$$

$$J_1^2 + J_2^2 = \vec{J}^2 - J_3^2$$

$$H_{sym} = \frac{1}{2I}\vec{J}^2 + \left(\frac{1}{2I_3} - \frac{1}{2I}\right) J_3^2$$

in a basis $|j, m\rangle$ we have

$$H_{sym}|j,m\rangle = \frac{\hbar^2}{2I} j(j+1) + \hbar^2\left(\frac{1}{2I_3} - \frac{1}{2I}\right) m^2$$

Since m appears only as m^2

j, m	$m = 0$	degeneracy	1
	$m \neq 0$	"	2

b)

$$H = \frac{1}{2}\quad \frac{1}{I+\Delta} J_1^2 + \frac{1}{I-\Delta} J_2^2 + \frac{1}{2I_3} J_3^2$$

since we want to first order in Δ expansion to first order in Δ is enough

$$H = H_{sym} - \frac{\Delta}{2I^2}(J_1^2 - J_2^2)$$

$$H_1 = -\frac{\Delta}{2I^2}(J_1^2 - J_2^2)$$

$$= -\frac{\Delta}{4I^2}(J_+^2 + J_-^2)$$

since $J_+|0,0\rangle = J_-|0,0\rangle = 0$

we have

$\langle 0|H_1|0\rangle = 0$ and the energy shift

vanishes.

c) From a) we see that

$|j=1, m=0\rangle$ is non degenerate

and we can use non degenerate perturbation theory.

but

$|j=1, m=\pm 1\rangle$ are degenerate

Furthermore H_1 is not diagonal in the subspace $|j=1, m=\pm 1\rangle$ and hence one has to use degenerate pertubation

Since $J_+^2 |j=1, m=-1\rangle \sim |j=1, m=+1\rangle$

$J_-^2 |j=1, m=1\rangle \sim |j=1, m=-1\rangle$

d) for $|j=1, m=0\rangle$ state

$$\langle j=1, m=0 | H_1 | j=1, m=0\rangle = 0$$

hence engy shift vanishes

for $|j=1, m=\pm 1\rangle$

Matrix element:

$$\langle j=1, m=+1 | H_1 | j=1, m=-1\rangle$$

$$= \langle j=1, m=+1 | -\frac{\Delta}{4I^2} J_+^2 | j=1\ m=-1\rangle$$

$$= -\frac{\Delta}{4I^2} 2\hbar^2 \qquad \text{using } (0.4)$$

Eigenvalues $\det\begin{pmatrix} \Delta E & -\frac{\Delta\hbar^2}{2I^2} \\ -\frac{\Delta\hbar^2}{2I^2} & \Delta E \end{pmatrix}$

$$\Rightarrow \boxed{\Delta E = \pm \frac{\Delta\hbar^2}{2I^2}}$$

4b)

$$c_n(t) = c_n(t_i) - \frac{i}{\hbar}\int_{t_i}^{t} dt' \sum_m \langle n|V_I(t')|m\rangle c_m(t')$$

$$V_I = e^{\frac{it}{\hbar}H_0} V_S\, e^{-\frac{it}{\hbar}H_0}$$

$$\Rightarrow \quad c_n(t) = c_n(t_i) - \frac{i}{\hbar}\int_{t_i}^{t} dt' \sum_m e^{-\frac{it}{\hbar}(E_m - E_n)} \langle n|W|m\rangle$$

for 2p→1s transition:

$$c_{2p}(t=+\infty) = -\frac{i}{\hbar}\int_{-\infty}^{+\infty} dt' \frac{B\tau}{\pi(t'^2+\tau^2)} \cdot e^{-\frac{it'}{\hbar}(E_1 - E_2)}$$

$$\langle n|V|m\rangle$$

$$E_n = -13.6\,\text{eV}\,\frac{Z}{n^2}$$

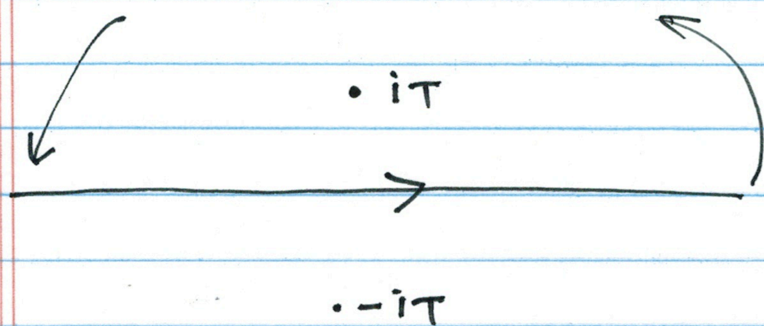
$$E_1 - E_2 = \Delta E < 0$$

Integral:

$$\frac{B\tau}{\pi}\int dt' \, \frac{1}{t'^2+\tau^2}\, e^{-\frac{it'}{\hbar}\Delta E} = B\, e^{+\frac{\Delta E \tau}{\hbar}}.$$

Can be obtaned by contor deformation.

since $\Delta E<0$ close contor in upper half plane



$$\frac{B\tau}{\pi}\oint \frac{-1}{2i\tau}\left(\frac{1}{t'+i\tau}-\frac{1}{t'-i\tau}\right) e^{-\frac{it'\Delta E}{\hbar}}$$

$$= + \frac{B}{2\pi i}\left(\overset{2\pi i}{+} \mathrm{Res}\,(t'=i\tau)\, e^{-\frac{it'\Delta E}{\hbar}} \right)$$

2 minus signs one from $+\frac{1}{t'-i\tau}$

$$= B\, e^{\frac{\Delta E \tau}{\hbar}}$$

~~one then contour is added~~

4b) corrected.

$\langle 2p | z | 1s \rangle$

$$= \int dr\, r^2 \int d\theta \sin\theta\, d\varphi\; R_{21} Y_1^0 \sqrt{\frac{4\pi}{3}}\, r\, Y_1^0\, R_{10} Y_0$$

$$= \frac{1}{\sqrt{3}} \int d\theta\, d\varphi \sin\theta\; Y_1^0 Y_1^0 \int dr\, r^3 R_{21} R_{10}$$

$$= \frac{1}{\sqrt{3}} \int dr\, r^3\; R_{21} R_{10}$$

$$= \frac{1}{\sqrt{3}} \frac{Z^4}{a_0^4} \frac{1}{2^{3/2}} \frac{2}{\sqrt{3}} \int_0^\infty dr\, r^4 e^{-\frac{3Zr}{2a_0}}$$

$$= \frac{1}{3\, 2^{1/2}} \frac{Z^4}{a_0^4} \left(\frac{2}{3}\frac{a_0}{Z}\right)^5 \int dx\, x^4 e^{-x}$$

$$= \frac{128\sqrt{2}}{343} \frac{a_0}{Z}.$$

$\Rightarrow$ prob: $|c_{2p}|^2 = \frac{B^2}{\hbar^2} e^{\frac{\Delta E T}{\hbar}} \cdot \left(\frac{128\sqrt{2}}{343}\frac{a_0}{2}\right)^2$

PHYSICS 221A

Practice Midterm – Fall 2013

Real midterm Monday Nov 4th, 2013, at 9.10am - 10.40am, in class

- Please write clearly
- Print your name on every page used, including this one;
- Make clear which question and which part you are answering on each page.
- No core-dumps please !
- No books, notes, computers, or calculators are allowed during the exam;
- Please turn off all electronic devices.

Good Luck !!

question	possible points	achieved points
1.	20	
2.	40	
3.	40	
Total	100	

Some possibly useful formulas

- The Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{0.1}$$

- Harmonic oscillator

$$[a, a^\dagger] = 1 \tag{0.2}$$

$$a = \frac{1}{\sqrt{2m\omega\hbar}}\Big(m\omega X + iP), \qquad a^\dagger = \frac{1}{\sqrt{2m\omega\hbar}}\Big(m\omega X - iP), \tag{0.3}$$

- Position and momentum

$$[X, P] = i\hbar \tag{0.4}$$

QUESTION 1: Quick questions [20 points]

a) For a linear operator A how are the following expectation values related ? († denotes the hermitian conjugate).

$$\langle \psi_1 \mid A \mid \psi_2 \rangle, \qquad \langle \psi_2 \mid A^\dagger \mid \psi_1 \rangle \tag{0.5}$$

b) State the generalized Heisenberg uncertainty principle for two hermitian observables A, B.

c) For the harmonic oscillator the energy eigenstates are given by $\mid n\rangle = 1/\sqrt{n!}(a^\dagger)^n \mid 0\rangle$. For which n does the matrix element

$$\langle 0 \mid \hat{x} \mid n \rangle \tag{0.6}$$

vanish ?

d) Which of these 2×2 matrices could be density matrices of a 2 state system ?

$$\begin{pmatrix} \frac{2}{3} & 0 \\ 0 & \frac{2}{3} \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} \frac{1}{3} & 1 \\ -1 & \frac{2}{3} \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} \frac{2}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \tag{0.7}$$

e) Which of the following operators is a possible observable ? (Here $\hat{x}$ and $\hat{p}$ are position and momentum operator satisfying $[x, p] = i\hbar$ and a are real constant).

$$
\begin{array}{lc}
1.) & \hat{x}\hat{p}, \\
2.) & (\hat{x} + ia\hat{p})(\hat{x} - ia\hat{p}), \\
3.) & p^2 + ax^2 \\
4.) & e^{iap}
\end{array}
$$

- extra space -

QUESTION 2: [40 points]

Consider a physical system with a three dimensional Hilbert space spanned by the vectors

$$\mid u_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mid u_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mid u_3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{0.8}$$

The Hamilton operator of the system is given by

$$H = \hbar\omega \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} \tag{0.9}$$

Two additional observables are given by

$$A = a \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad B = b \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{0.10}$$

At time $t = 0$ the system is in the following state:

$$\mid \psi(t=0)\rangle = \frac{1}{\sqrt{2}} \mid u_1\rangle + \frac{1}{2} \mid u_2\rangle + \frac{1}{2} \mid u_3\rangle \tag{0.11}$$

a) At time $t = 0$ we measure the energy of the system. What are the possible values one can obtain and related probabilities ? Calculate the expectation value and the variance of the energy.

b) Instead of measuring the energy we measure the observable A. What are the possible values one can obtain and what are the probabilities ? Determine the state of the system after the measurement of A (for each possible result of the measurment).

c) Instead of measuring anything, the system in state (0.11) evolves from time $t = 0$ according to the Schrödinger equation. Determine the state $\mid \psi(t)\rangle$ at time t and calculate the expectation values for A and B at time t. How do they differ ?

d)

We now measure the observable A at time t. What are the possible values one can obtain and the probabilities ? Same question if one measures B instead at time t.

- extra space -

- extra space -

- extra space -

QUESTION 3: [40 points]

A finite dimensional lattice with periodic boundary conditions has a Hilbert space spanned by $\mid n\rangle, n = 1, 2, \cdots N$, where

$$\langle n|m\rangle = \delta_{mn} \tag{0.12}$$

At the lattice is made periodic by the identification $\mid N+1\rangle \equiv \mid 1\rangle$. The expectation values of the Hamiltonian are given by

$$\begin{aligned}
\langle n \mid H \mid n\rangle &= r_0 \\
\langle n \mid H \mid n+1\rangle &= r_1 e^{i\phi_1} \\
\langle n \mid H \mid n+2\rangle &= r_2 e^{i\phi_2}
\end{aligned}$$

Where $r_0, r_1, r_2, \phi_1, \phi_2$ are real.

a) Calculate the expectation values $\langle n+1 \mid H \mid n\rangle$ and $\langle n+2 \mid H \mid n\rangle$

b) Express the the Hamiltonian in term of the translation operator T which satisfies

$$T \mid n\rangle = \mid n+1\rangle, \quad T^\dagger \mid n\rangle = \mid n-1\rangle, \tag{0.13}$$

c) Find the eigenstates and eigenvalues of H.

d) For the position operator X the following relation holds

$$X \mid n\rangle = a\, n \mid n\rangle \tag{0.14}$$

where a is a called the lattice spacing. Show that X is hermitan and that it satisfies the operator identity

$$T^\dagger X T = a + X \tag{0.15}$$

- extra space -

- extra space -

- extra space -

221A Practice Midterm Solution

Antonio Russo
October 31, 2013

1 Quick Questions

(1a) Hermitian Conjugate

$$\langle\psi_1| A |\psi_2\rangle = \overline{\langle\psi_2| A^\dagger |\psi_1\rangle}$$

(1b) Heienberg Uncertainty

$$(\Delta_\phi A)^2(\Delta_\phi B)^2 \geq \frac{1}{4}\left|\langle[A,B]\rangle_\phi\right|^2$$

(1c) Ladder Operators

Since $\hat{x} \sim a + a^\dagger$, this term vanishes for $n \neq 1$.

(1d) Density Matrices

The first has trace of 4/3, so no. The second has trace 1, and is hermitian, but has a negative eigenvalue (so no). The third is not hermitian (so no). The fourth is just a projector (so yes). The last has trace 1, is hermitian, and is positive semi-definite (so yes).

(1e) Observable

Check if they are hermitian. For (1), because $\hat{x}\hat{p} \neq \hat{p}\hat{x}$, (1) is not an observable. The second and third are, while the fourth is not (because of the i).

2

(2a) Energy at $t = 0$

$$\begin{aligned}\langle E\rangle &= \left[\frac{1}{\sqrt{2}}\langle u_1| + \frac{1}{2}\langle u_2| + \frac{1}{2}\langle u_3|\right] H \left[\frac{1}{\sqrt{2}}|u_1\rangle + \frac{1}{2}|u_2\rangle + \frac{1}{2}|u_3\rangle\right] \\ &= \hbar\omega\left[\frac{1}{\sqrt{2}}\langle u_1| + \frac{1}{2}\langle u_2| + \frac{1}{2}\langle u_3|\right]\left[\frac{1}{\sqrt{2}}|u_1\rangle + \frac{3}{2}|u_2\rangle + \frac{3}{2}|u_3\rangle\right] = \hbar\omega\left[\frac{1}{2} + \frac{3}{4} + \frac{3}{4}\right] = 2\hbar\omega\end{aligned}$$

and

$$\begin{aligned}\left\langle E^2\right\rangle &= \left[\frac{1}{\sqrt{2}}\langle u_1| + \frac{1}{2}\langle u_2| + \frac{1}{2}\langle u_3|\right] H^2 \left[\frac{1}{\sqrt{2}}|u_1\rangle + \frac{1}{2}|u_2\rangle + \frac{1}{2}|u_3\rangle\right] \\ &= (\hbar\omega)^2\left[\frac{1}{\sqrt{2}}\langle u_1| + \frac{1}{2}\langle u_2| + \frac{1}{2}\langle u_3|\right]\left[\frac{1}{\sqrt{2}}|u_1\rangle + \frac{9}{2}|u_2\rangle + \frac{9}{2}|u_3\rangle\right] = (\hbar\omega)^2\left[\frac{1}{2} + \frac{9}{4} + \frac{9}{4}\right] = 5(\hbar\omega)^2\end{aligned}$$

The variance is then

$$\left\langle E^2\right\rangle - \langle E\rangle^2 = (\hbar\omega)^2[5-4] = (\hbar\omega)^2$$

One can obtain $E = \hbar\omega$ with probability $\frac{1}{2}$, and $E = 3\hbar\omega$ with probability $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.

(2b) A at $t = 0$

Notice that $A|\psi\rangle = a|\psi\rangle$. We obtain a with probability 1, and we are in the same state as we were before.

(2c) $t > 0$

Notice that
$$|\psi(t)\rangle = \frac{1}{\sqrt{2}} e^{-i\omega t} |u_1\rangle + \frac{1}{2} e^{-3i\omega t} |u_2\rangle + \frac{1}{2} e^{-i3\omega t} |u_3\rangle$$
This is still an eigenstate of A with eigenvalue a, so the expectation value of A is a (more generally, since $[A, H] = 0$, we would always have that values related to A would be unchanged). For B,
$$B|\psi(t)\rangle = \frac{b}{2} e^{-3i\omega t} |u_1\rangle + \frac{b}{\sqrt{2}} e^{-i\omega t} |u_2\rangle + \frac{b}{2} e^{-i3\omega t} |u_3\rangle$$
so that
$$\langle B\rangle = \frac{b}{2\sqrt{2}} e^{-2i\omega t} + \frac{b}{2\sqrt{2}} e^{2i\omega t} + \frac{b}{4} = b\left[\frac{\cos(2\omega t)}{\sqrt{2}} + \frac{1}{4}\right]$$

(2d) A and B

We have already discussed A. B has eigenvectors
$$B|u_3\rangle = b|u_3\rangle \quad \text{and} \quad B\frac{1}{\sqrt{2}}[|u_1\rangle + |u_2\rangle] = b\frac{1}{\sqrt{2}}[|u_1\rangle + |u_2\rangle] \quad \text{and} \quad B\frac{1}{\sqrt{2}}[|u_1\rangle - |u_2\rangle] = -b\frac{1}{\sqrt{2}}[|u_1\rangle - |u_2\rangle]$$
Looking at $|\psi(t)\rangle$, we see that we can have $-b$ with probability
$$\begin{aligned}
&\left|\frac{1}{\sqrt{2}}[\langle u_1| - \langle u_2|]\left[\frac{1}{\sqrt{2}} e^{-i\omega t} |u_1\rangle + \frac{1}{2} e^{-3i\omega t} |u_2\rangle + \frac{1}{2} e^{-i3\omega t} |u_3\rangle\right]\right|^2 \\
&\qquad = \left|\frac{1}{2} e^{-i\omega t} - \frac{1}{2\sqrt{2}} e^{-3i\omega t}\right|^2 = \frac{1}{4}\left|1 - \frac{1}{\sqrt{2}} e^{-2i\omega t}\right|^2 \\
&\qquad = \frac{1}{4}\left[1 - \frac{1}{\sqrt{2}} e^{-2i\omega t}\right]\left[1 - \frac{1}{\sqrt{2}} e^{2i\omega t}\right] = \frac{1}{4}\left[1 - \frac{2}{\sqrt{2}}\cos(2\omega t) + \frac{1}{2}\right]
\end{aligned}$$
And we avoid doing any more work and conclude that we can get b with probability
$$1 - \frac{1}{4}\left[1 - \frac{2}{\sqrt{2}}\cos(2\omega t) + \frac{1}{2}\right]$$

3

We use the shorthand
$$A_i = r_i e^{i\alpha_i}$$

(3a)

$$\langle n+1| H |n\rangle = \left[\langle n| H^\dagger |n+1\rangle\right]^* = [\langle n| H |n+1\rangle]^* = A_1^* = r_1 e^{-i\alpha_1}$$
and
$$\langle n+2| H |n\rangle = \left[\langle n| H^\dagger |n+2\rangle\right]^* = [\langle n| H |n+2\rangle]^* = A_2^* = r_2 e^{-i\alpha_2}$$

(3b)

$$H = A_0 + A_1 T_{-1} + A_1^* T_1 + A_2 T_{-2} + A_2^* T_2$$

(3c)

Suppose that

$$|k\rangle = \sum_n e^{-ikn} |n\rangle$$

(where $k = \frac{2\pi m}{N}$, for some $0 \leq m < N$). Then,

$$T |k\rangle = \sum_n e^{-ikn} |n+1\rangle = e^{ik} \sum_n e^{-ikn} |n\rangle$$

then,

$$H |k\rangle = \left[A_0 + A_1 e^{-ik} + A_1^* e^{ik} + A_2 e^{-2ik} + A_2^* e^{2ik}\right] |k\rangle = [A_0 + 2r_1 \cos(\alpha_1 - k) + 2r_2 \cos(\alpha_2 - 2k)] |k\rangle$$

(3d)

$$\langle m| X^\dagger |n\rangle = [\langle n| X |m\rangle]^* = [\langle n| am |m\rangle]^* = am\delta_{nm}$$

which is plainly the same as

$$\langle m| X |n\rangle = \langle m| an |n\rangle = an\delta_{nm}$$

Since this holds for all n, X is hermitian. Next,

$$T^\dagger XT |n\rangle = T^\dagger X |n+1\rangle = a(n+1) T^\dagger |n+1\rangle = (n+1) |n\rangle = a |n\rangle + an |n\rangle = (a + X) |n\rangle$$

so, because n is arbitrary, we have the claim.

Name: UID:

PHYSICS 221A

Midterm – Fall 2013

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Good Luck !!

question	possible points	achieved points
1.	20	
2.	40	
3.	40	
Total	100	

Some possibly useful formulas

- The Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{0.1}$$

- Harmonic oscillator

$$[a, a^\dagger] = 1 \tag{0.2}$$

$$a = \frac{1}{\sqrt{2m\omega\hbar}}\big(m\omega X + iP\big), \qquad a^\dagger = \frac{1}{\sqrt{2m\omega\hbar}}\big(m\omega X - iP\big), \tag{0.3}$$

- Position and momentum

$$[X, P] = i\hbar \tag{0.4}$$

QUESTION 1: Quick questions [20 points]

a) For a three state system and an orthonormal basis $\langle i \mid j\rangle = \delta_{ij}, i, j = 1, 2, 3$. Which of the following operators are observables ?

$$\begin{aligned} O_1 &= 2\epsilon_0 \mid 1\rangle\langle 1 \mid +2\epsilon_0 \mid 2\rangle\langle 2 \mid +3\epsilon_0 \mid 3\rangle\langle 3 \mid & (0.5)\\ O_2 &= \epsilon_0 \mid 1\rangle\langle 2 \mid -\epsilon_0 \mid 2\rangle\langle 1 \mid & (0.6)\\ O_3 &= \epsilon_0 \mid 1\rangle\langle 1 \mid -\epsilon_1 \mid 2\rangle\langle 3 \mid -\epsilon_1 \mid 3\rangle\langle 2 \mid & (0.7)\end{aligned}$$

O_1 & O_3 are hermitian & observables

b) True or false in general ? For two operators commute, i.e. $[A, B] = 0$: If $\mid a\rangle$ is an eigenvector of A, it is automatically an eigenvector of B.

Not true counter example 3×3 matrices

$A = \begin{pmatrix} 1 & & \\ & 2 & \\ & & 2 \end{pmatrix}$ $B = \begin{pmatrix} 2 & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix}$ & $|a\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

c) What equation does an operator O satisfy if it is 1. a projector, 2. a hermitian operator, 3. a unitary operator ?

1. $O^2 = O$

2. $O^+ = O$ or $\langle \phi | O | \psi \rangle = \langle \psi | O | \phi \rangle^* \quad \forall \varphi, \psi$

3. $O^{-1} = O^+$ or $O^+ O = O O^+ = \mathbb{1}$

d) For a two state system describe by a density matrix

$$\rho = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{2}{3} \end{pmatrix} \tag{0.8}$$

Calculate the expectation value of the operator $S_i = \frac{1}{2}\hbar\sigma_i, i = x, y, z$ where σ_i are the Pauli matrices.

$\mathrm{tr}\, \rho\, S_i$

$\mathrm{tr}\, \rho\, S_z = \frac{1}{2}\hbar\ \mathrm{tr}\begin{pmatrix} 1/3 & \\ & 2/3 \end{pmatrix}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} = \frac{1}{2}\hbar\left(\frac{1}{3} - \frac{2}{3}\right) = -\hbar\,\frac{1}{6}$

$\mathrm{tr}(\rho\, S_y) = \frac{1}{2}\hbar\ \mathrm{tr}\begin{pmatrix} 1/3 & \\ & 2/3 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = 0$

$\mathrm{tr}\, \rho\, S_x = \frac{1}{2}\hbar\ \mathrm{tr}\begin{pmatrix} 1/3 & \\ & 2/3 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 0$

e) State briefly (following the Copenhagen interpretation), the two ways a wave function can evolve in time.

1. Evolution via Schrödinger equation

$i\hbar \frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle$ no measurement

2. Reduction of the wave funct

Measure Observable A & obtain Eigenvalue a

$|\psi\rangle \to P(a)|\psi\rangle$

$P(a)$ projector onto Eigenspace of a

- extra space - Please state which problem you are working on !

QUESTION 2: [40 points]

a) Consider the operator

$$T(a) = \exp(-iaP/\hbar) \tag{0.9}$$

Where a is a constant and P is the momentum operator Show that

$$T^\dagger(a)\ X\ T(a) = x + a \tag{0.10}$$

$[X,P]=i\hbar$ use $e^{+A}Be^{-A} = \exp \mathrm{Ad}_A B$

$$e^{+iaP/\hbar} X e^{-iaP/\hbar} = \left(1 + \mathrm{ad}_{iaP/\hbar}\right)X$$

$$= X + i a/\hbar\, [P,X]$$

$$= X + a$$

b) Show that $T(a)$ is unitary and show that T(a) has eigenvalues of of the form $e^{i\phi}$ where ϕ is real (You can assume that P is hermitian).

$T(a)^\dagger T(a) = e^{iaP/\hbar} e^{-iaP/\hbar} = 1$ since P commutes with itself

same for $T(a)T(a)^\dagger \Rightarrow T$ unitary

$\langle\psi|T(a)|\psi\rangle = \langle\psi|\lambda|\psi\rangle = \lambda$

$\langle\psi|T(a)|\psi\rangle^* = \langle\psi|T(a)^\dagger|\psi\rangle = \lambda^*$

on the other hand

$|\psi\rangle = \lambda\, T(a)^\dagger|\psi\rangle \Rightarrow \lambda^* = \frac{1}{\lambda}$ $\quad \lambda\lambda^* = 1$ $\Rightarrow \lambda = e^{i\phi}$

c) Consider the Hamiltonian which is periodic under shifts by a post quest.

$$H = \frac{P^2}{2m} + \sum_{n=-\infty}^{\infty} V(\cancel{X} - na) \tag{0.11}$$

Here you can assume that $V(x)$ goes exponentially fast to zero as $|x| \to \infty$ (This assumption makes the sum over n convergent). You can also assume that $V(x)$ can be expanded in a power series.

Prove that $T(a)$ commutes with H.

$$V(x) = \sum_n c_n x^n$$

$$T^\dagger X T = X + a$$

$$T^\dagger X^n T = T^\dagger X T T^\dagger \dots T T^\dagger X T$$

$$= (X+a)^n$$

$$\Rightarrow \quad T^\dagger V(x) T = \sum_n c_n (X+a)^n$$

$$= V(x+a)$$

$$T^\dagger \sum_{n=-\infty}^{\infty} V(X-na) T = \sum_{n=\infty}^{\infty} V(X+a-na)$$

$$= \sum_n V(x-(\underbrace{n-1}_{n'})a)$$

$$= \sum_{n'} V(x-n'a) = \sum V(X-na)$$

$$T^\dagger P^2 T = P^2$$

hue $\quad T^\dagger H T = H \quad \sim \quad HT = TH \Leftrightarrow [T, H] = 0$

d) It follows from the results in part c) that the Hamiltonian H and $T(a)$ can be diagonalized simultaneously. You can assume that there are eigenstates $\mid E, k\rangle$ which satisfy

$$\begin{aligned} H \mid E, k\rangle &= E \mid E, k\rangle & (0.12)\\ T(a) \mid E, k\rangle &= e^{-ika} \mid E, k\rangle & (0.13) \end{aligned}$$

For the wave functions in position space define the following combination

$$u_k(x) = \langle x \mid E, k\rangle e^{-ikx} \tag{0.14}$$

Show that $u_k(x)$ is a periodic function with period a, i.e.

$$u_k(x+a) = u_k(x) \tag{0.15}$$

This is the Bloch's theorem for periodic potentials (i.e. an energy eigenstate can be written as a Bloch wave times a periodic function).

$$u_k(x+a) = \langle x+a \mid E k\rangle e^{-ik(x+a)}$$

$$= (\langle E k \mid x+a\rangle)^* e^{-ik(x+a)}$$

$$= \langle E, k \mid T \mid x\rangle^* e^{-ik(x+a)}$$

$$= \langle x \mid T^+ \mid E, k\rangle e^{-ikx} e^{-ika}$$

$$= \langle x \mid E, k\rangle \cdot e^{ika} e^{-ikx} \cancel{e^{-ika}}$$

$$= u_k(x)$$

$\underline{\text{Use}}$ $T^+_{(a)} T(a) \mid E k\rangle = e^{-ika} T^+ \mid E k\rangle$

$\to T^+ \mid E, k\rangle = e^{ika} \mid E, k\rangle$

QUESTION 3: [40 points]

a) For the ladder operators

$$a = \frac{1}{\sqrt{2m\omega\hbar}}(m\omega X + iP), \qquad a^\dagger = \frac{1}{\sqrt{2m\omega\hbar}}(m\omega X - iP), \tag{0.16}$$

Using the basic commutation relation for X and P, calculate

$$[a^\dagger, a] \tag{0.17}$$

$$[a^+, a] = \frac{1}{2m\omega\hbar}[m\omega X - iP, m\omega X + iP]$$

$$= \frac{1}{2m\omega\hbar}\Big(m\omega i\,[X, P] - m\omega i\,[P, X]\Big)$$

$$= \frac{-2i\hbar}{2\hbar}$$

$$= -1$$

b) Express the following operators in terms of a and $a^\dagger$

1.	X	(0.18)
2.	P	(0.19)
3.	X^2	(0.20)
4.	P^2	(0.21)

$$a = \frac{1}{\sqrt{2m\omega\hbar}}(m\omega X + iP)$$

$$a^\dagger = \frac{1}{\sqrt{2m\omega\hbar}}(m\omega X - iP)$$

$$a + a^\dagger = \frac{2m\omega}{\sqrt{2m\omega\hbar}} X$$

$$\Rightarrow X = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger)$$

$$a - a^\dagger = \frac{i2}{\sqrt{2m\omega\hbar}} P$$

$$\Rightarrow P = \frac{1}{i}\sqrt{\frac{m\omega\hbar}{2}}(a - a^\dagger)$$

$$X^2 = \frac{\hbar}{2m\omega}(a + a^\dagger)^2$$

$$P^2 = -\frac{m\omega\hbar}{2}(a - a^\dagger)^2$$

c) Calculate the expectation value of X and P for the system in the energy eigenstate $| 1\rangle = a^\dagger | 0\rangle$

$$\langle 1 | X | 1 \rangle = 0$$ because.

$$= \langle 0 | a \quad X a^\dagger | 0 \rangle$$

$$= \langle 0 | a \sqrt{\frac{\hbar}{2m\omega}} (a + a^\dagger) \; a^\dagger | 0 \rangle$$

$$\langle 0 | a^2 a^\dagger | 0 \rangle = 0 \langle 0 | a | 0 \rangle$$

$$\langle 0 | a \; a^{\dagger 2} | 0 \rangle = \langle 0 | a^\dagger | 0 \rangle$$

$$\langle 1 | P | 1 \rangle = \langle 0 | a \; P a^\dagger | 0 \rangle$$

$$= -i \sqrt{\frac{\hbar m \omega}{2}} \langle 0 | a (a - a^\dagger) a^\dagger | 0 \rangle$$

$$= 0$$ same care

d) Calculate the Expectation value for X^2 and P^2 for the system in the energy eigenstate $\mid 1\rangle = a^\dagger \mid 0\rangle$ and check wether the Heisenberg uncertainty relation for X and P is saturated for the state $\mid 1\rangle = a^\dagger \mid 0\rangle$.

$$\langle 1 | X^2 | 1 \rangle$$

$$= \langle 0 | a \frac{\hbar}{2m\omega} (a + a^\dagger)^2 a^\dagger | 0 \rangle$$

$a^3 a^\dagger$ & $a a^{\dagger 3}$ are zero

$$= \frac{\hbar}{2m\omega} \langle 0 | a (a a^\dagger + a^\dagger a) a^\dagger | 0 \rangle$$

$$= \frac{\hbar}{2m\omega} \langle 0 | a (2N + 1) a^\dagger | 0 \rangle$$

$$= \frac{\hbar}{2m\omega} \, 3 \langle 0 | 0 \rangle = \frac{3\hbar}{2m\omega}$$

$$\langle 1 | P^2 | 1 \rangle = -\frac{m\omega\hbar}{2} \langle 0 | a (a - a^\dagger)^2 a^\dagger | 0 \rangle$$

$$= \frac{m\omega\hbar}{2} \langle 0 | a (a a^\dagger + a^\dagger a) a^\dagger | 0 \rangle$$

$$= \frac{3 m\omega\hbar}{2}$$

$$(\Delta_\psi X)^2 (\Delta_\psi P)^2 = \langle 1 | X^2 | 1 \rangle \langle 1 | P^2 | 1 \rangle = \frac{9}{4} \hbar^2$$

$$\frac{1}{4} \left| \langle 1 | [X, P] | 1 \rangle \right|^2 = \frac{\hbar^2}{4} \langle 1 | 1 \rangle = \frac{\hbar^2}{4}$$

not saturated

Name: UID:

PHYSICS 221A

Practice Final

Real Exam: Monday Dec 9th 3-6pm, room: PAB 2-434

- Please write clearly
- Please write down your name and UID on the front page, if you separate sheets please write your name on each sheet.
- Make clear which question and which part you are answering on extra each page
- No core-dumps please !
- No books, notes, computers, or calculators are allowed during the exam;
- Please turn off all electronic devices.
- All parts of questions, a),b)c) etc., carry equal weight unless otherwise indicated.

question	possible points	achieved points
1.	40	
2.	40	
3.	40	
4.	40	
5.	40	
Total	200	

Some possibly useful formulas

• The angular momentum algebra is given by $[J_1, J_2] = i\hbar J_3$, and cyclic permutations. The ladder operators, defined by $J_\pm \equiv J_1 \pm iJ_2$, act as follows,

$$J_\pm |j,m\rangle = \hbar\sqrt{j(j+1) - m(m\pm 1)}\,|j, m\pm 1\rangle \tag{0.1}$$

where the states are properly normalized by $\langle j', m'|j,m\rangle = \delta_{j,j'}\delta_{m,m'}$.

• "little" CBH-formula

$$e^A B e^{-A} = \exp(Ad_A)B \tag{0.2}$$

where $Ad_X \cdot = [X, \cdot]$.

• The Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{0.3}$$

• harmonic oscillator

$$\begin{aligned} a &= \frac{1}{\sqrt{2m\omega\hbar}}\Big(m\omega\, X + i\, P\Big) \\ a^\dagger &= \frac{1}{\sqrt{2m\omega\hbar}}\Big(m\omega\, X - i\, P\Big) \end{aligned} \tag{0.4}$$

• Position/momentum

$$[X, P] = i\hbar \tag{0.5}$$

QUESTION 1: Short questions [40 points]

a) For an operator A (with no explicit time dependence) in the Heisenberg picture, what is the condition it has to satisfy for it to be conserved, i.e.

$$\frac{d}{dt}A = 0 \tag{0.6}$$

b) If you add a two angular momenta with $(\vec{J}_1)^2 = 30\hbar^2$ and $(\vec{J}_2)^2 = 20\hbar^2$, what are the possible values of the total angular momentum squared ?

c) A gauge transformation is given by

$$\vec{A} \to \vec{A} + \vec{\nabla}\theta, \qquad \psi \to e^{i\frac{e\theta}{\hbar}}\psi \tag{0.7}$$

Where $\theta(x)$ is an arbitrary function of $\vec{x}$. Show that the covariant derivative of the wave function

$$\Big(- i\hbar\vec{\nabla} - e\vec{A}\Big)\psi(x) \tag{0.8}$$

transforms in the same way as ψ.

d) State the Ehrenfest theorem.

e) For a particle of spin 1/2 moving in a three dimensional central potential, list a maximal set of commuting observables.

f) For a matrix U in $SU(N)$ the matrix has to satisfy

$$U^\dagger U = 1, \qquad \det(U) = 1 \tag{0.9}$$

What does this imply for the infentisimal generator T ?

$$U = 1 + i\,\epsilon\,T + o(\epsilon^2) \tag{0.10}$$

g) You have three observables A, B, C they satisfy

$$[A, B] = 0, \qquad [A, C] = 0 \tag{0.11}$$

what can you say about $[A, [B, C]]$? (Justify your answer).

h) For the classical quantity $x^2 p^2$ write down two possible forms of the corresponding observable according to the correspondence principle.

QUESTION 2: [40 points]

A system of three (non-identical) spin 1/2 particles, whose spin operators are $\vec{S}_1, \vec{S}_2, \vec{S}_3$, is governed by the Hamiltonian,

$$H = \frac{2A}{\hbar^2}\vec{S}_1 \cdot \vec{S}_2 + \frac{2B}{\hbar^2}\vec{S}_3 \cdot (\vec{S}_1 + \vec{S}_2) \tag{0.12}$$

where A and B are real constants.

a) Show that $\vec{S}_1 + \vec{S}_2$ and $\sum_{i=1,2,3} \vec{S}_i$ bothe satisfy the commutation relation of spin/angular momentum.

b) What are the possible eigenvalues that $(\vec{S}_1 + \vec{S}_2)^2$ and $(\vec{S}_1 + \vec{S}_2 + \vec{S}_3)^2$ can take ?

c) [10pts] Rewrite the Hamiltonian such that in only involves squares of $\vec{S}_i$ or squares of sums of $\vec{S}_i$'s.

d) [15pts] Calculate the energy levels and their respective degeneracies.

Note: In part d) you can quote results of representation theory and addition of angular momentum

QUESTION 2: [40 points]

A coherent state for a single harmonic oscillator is given by

$$| c\rangle_{\text{coh}} = e^{\frac{-|c|^2}{2}} e^{ca^\dagger} \mid 0\rangle \tag{0.13}$$

Where $\mid 0\rangle$ is the ground state of the harmonic oscillator.

a) Show that $| c\rangle_{\text{coh}}$ is normalized

b) Are there any values of c_1, c_2 for which two coherent states are orthogonal ? (Back up your answer with an argument or a calculation).

c) Show that $| c\rangle_{coh}$ is an eigenstate of the lowering operator a and calculate the eigenvalue.

d) Show that

$$|\langle n \mid c\rangle_{coh}|^2 = \frac{A^n}{n!} e^{-B} \tag{0.14}$$

and determine A and B. Here

$$\mid n\rangle = \frac{1}{\sqrt{n!}} (a^\dagger)^n \mid 0\rangle \tag{0.15}$$

is the n-th excited state of the harmonic oscillator.

QUESTION 3: [40 points]

Consider a one dimensional particle moving in a potential with Hamiltonian

$$H = \frac{p^2}{2m} + V(x) \tag{0.16}$$

Assume that the Hamiltonian has a discrete non-degenerate spectrum (i.e. there are only bound states)

$$H \mid n\rangle = E_n \mid n\rangle, \quad E_n \neq E_m \text{ if } n \neq m \tag{0.17}$$

a) Show that

$$[[x, H], x] = \frac{\hbar^2}{m} \tag{0.18}$$

b) Show that the following "sum rule" holds (it's called that because you sum over all states).

$$\sum_m (E_m - E_n) \left| \langle n \mid x \mid m\rangle \right|^2 = \frac{\hbar^2}{2m} \tag{0.19}$$

Hint: insert a complete set of states in the appropriate place.

c) Calculate

$$\langle n \mid x \mid m\rangle \tag{0.20}$$

for the harmonic oscillator

d) Verify the sum rule for the harmonic oscillator

QUESTION 4: [40 points]

Consider two (distinguishable) spin 1/2 particles with Hamiltonian

$$H = h_0\Big(\sigma_x^{(1)} \otimes \sigma_x^{(2)} + \sigma_y^{(1)} \otimes \sigma_y^{(2)}\Big) \tag{0.21}$$

Where σ_i are the Pauli-matrices.

a) Show that

$$[\sigma_x^{(1)} \otimes \sigma_x^{(2)}, \sigma_y^{(1)} \otimes \sigma_y^{(2)}] = 0 \tag{0.22}$$

b) Calculate the spectrum of H

c) Calculate the state at time $t \mid \psi(t)\rangle$ with the initial condition

$$\mid \psi(t=0)\rangle = \mid j = \frac{1}{2}, m = \frac{1}{2}\rangle \otimes \mid j = \frac{1}{2}, m = \frac{1}{2}\rangle \tag{0.23}$$

Where m denotes the eigenvalue of σ_z.

d) For the state obtained in c) Calculate

$$\langle\psi(t) \mid \sigma_z \otimes \sigma_z \mid \psi(t)\rangle \tag{0.24}$$

221A Practice Final Solution

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December 3, 2013

1 Short Questions

(1a) Conserved Observable

A must commute with H, i.e., $[A, H] = 0$

(1b) Addition of Angular Momentum

We have

$$(J_1)^2 = 30\hbar^2 = 5 \cdot 6 \cdot \hbar^2 = j_1(j_1+1)\hbar^2 \quad \text{so} \quad j_1 = 5$$

and

$$(J_2)^2 = 20\hbar^2 = 4 \cdot 5 \cdot \hbar^2 = j_2(2_1+1)\hbar^2 \quad \text{so} \quad j_2 = 4$$

The sum can will have anywhere from $|j_1 - j_2| = 1$ to $j_1 + j_2 = 9$, or J^2 between $2\hbar^2$ and $90\hbar^2$.

(1c) Gauge Transformation

Under the gauge transformation,

$$\begin{aligned}
\left[-i\hbar\vec{\nabla} - e\vec{A}\right]\psi \to \left[-i\hbar\vec{\nabla} - e\vec{A}'\right]\psi' &= \left[-i\hbar\vec{\nabla} - e\left(\vec{A} + \vec{\nabla}\theta\right)\right] e^{i\frac{e\theta}{\hbar}}\psi \\
&= \left[-e\left(\vec{A} + \vec{\nabla}\theta\right)\right] e^{i\frac{e\theta}{\hbar}}\psi - i\hbar\left[e^{i\frac{e\theta}{\hbar}}\vec{\nabla}\psi + i\frac{e}{\hbar}\left(\vec{\nabla}\theta\right)e^{i\frac{e\theta}{\hbar}}\psi\right] \\
&= e^{i\frac{e\theta}{\hbar}}\left[-i\hbar\vec{\nabla} - e\left(\vec{A} + \vec{\nabla}\theta\right)\right]\psi + e\left(\vec{\nabla}\theta\right)e^{i\frac{e\theta}{\hbar}}\psi = e^{i\frac{e\theta}{\hbar}}\left[-i\hbar\vec{\nabla} - e\vec{A}\right]\psi
\end{aligned}$$

(1d) Ehrenfest Theorem

$$\partial_t \langle A \rangle_\psi = \frac{1}{i\hbar}\langle [A, H] \rangle_\psi + \langle \partial_t A \rangle_\psi$$

(1e) Spin $1/2$ Maximal Set of commuting oberservables

Here is one such list: p_x, p_y, p_z, S_z (and, if you like, the Casimir operator S^2).

(1f) Infinitesimal Generator of Unitary Operator

If $U^\dagger = U^{-1}$, the tayor expansions

$$U^\dagger = 1 - i\epsilon T^\dagger + o(\epsilon^2)$$

and

$$U^{-1} = 1 - i\epsilon T + o(\epsilon^2)$$

are equal. Therefore $T = T^\dagger$ is hermitian.

(1g) Three Commutators

$$[A, [B, C]] = [A, BC] - [A, CB] = B[A, C] + [A, B]C - C[A, B] - [A, C]B = 0$$

(1h) Corresponding Quantum Operator

All symmetric combinations of x and p are candidate quantum operators.

$$x^2p^2 + p^2x^2, \quad ix^2p^2 - ip^2x^2, \quad xp^2x, \quad px^2p, \quad xpxp + pxpx, \quad ixpxp - ipxpx$$

2 Three Spins

(2a) Commutator

Since the commutators are linear, this follows immediately.

(2b) Eigenvalues

Put $\vec{J}_{12} = \vec{S}_1 + \vec{S}_2$ and $\vec{J}' = \vec{J}_{12} + \vec{S}_3$. Because

$$D^{1/2} \otimes D^{1/2} = D^0 \oplus D^1$$

j_{12} can be either 0 or 1. Similarly, because

$$\begin{aligned}\left(D^{(1/2)} \otimes D^{(1/2)}\right)_{12} \otimes D_3^{(1/2)} &= \left(D_{12}^{(0)} \oplus D_{12}^{(1)}\right) \otimes D_3^{(1/2)} = \left(D_{12}^{(0)} \otimes D_3^{(1/2)}\right) \oplus \left(D_{12}^{(1)} \otimes D_3^{(1/2)}\right) \\ &= D_{j_{12}=0}^{(1/2)} \oplus D_{j_{12}=1}^{(1/2)} \oplus D_{j_{12}=1}^{(3/2)}\end{aligned}$$

j' can be either $1/2$ or $3/2$. The possible eigenvalues are therefore

$$J_{12}^2 = 0, 2\hbar^2 \quad \text{and} \quad J'^2 = \frac{3}{4}\hbar^2, \frac{15}{4}\hbar^2$$

(we will use the full decomposition we found here in part (d).

(2c) Rewritten

Note that

$$J_{12}^2 = S_1^2 + 2\vec{S}_1 \cdot \vec{S}_2 + S_2^2 \quad \text{so that} \quad \vec{S}_1 \cdot \vec{S}_2 = \frac{1}{2}\left[J_{12}^2 - S_2^2 - S_1^2\right]$$

and

$$J'^2 = J_{12}^2 + 2\vec{J}_{12} \cdot \vec{S}_3 + S_3^2 \quad \text{so that} \quad \vec{S}_3 \cdot \vec{J}_{12} = \frac{1}{2}\left[J'^2 - J_{12}^2 - S_3^2\right]$$

It follows that

$$H = \frac{A}{\hbar^2}\left[J_{12}^2 - S_2^2 - S_1^2\right] + \frac{B}{\hbar^2}\left[J'^2 - J_{12}^2 - S_3^2\right]$$

or, collecting terms

$$\begin{aligned}H &= \frac{1}{\hbar^2}J_{12}^2\left[A - B\right] - \frac{A}{\hbar^2}\left[S_2^2 + S_1^2\right] - \frac{B}{\hbar^2}S_3^2 + \frac{1}{\hbar^2}J'^2 B \\ &= \frac{1}{\hbar^2}J_{12}^2\left[A - B\right] - \frac{A}{\hbar^2}\left[\frac{3}{4}\hbar^2 + \frac{3}{4}\hbar^2\right] - \frac{B}{\hbar^2}\frac{3}{4}\hbar^2 + \frac{1}{\hbar^2}J'^2 B \\ &= \frac{1}{\hbar^2}J_{12}^2\left[A - B\right] + \frac{1}{\hbar^2}J'^2 B - \frac{3}{2}A + \frac{3}{4}B\end{aligned}$$

(2d) Energies

The decomposition we performed in part (b) shows that j'^2 and j_{12}^2 are good quantum numbers. Indeed, when $j' = 1/2$, $j_{12} = 0, 1$. For $|\psi\rangle = \left|j' = 1/2, j_{12} = 0, j_z' = \pm 1/2\right\rangle$ (the 2 states)

$$H|\psi\rangle = \left(\frac{3}{4}B - \frac{3}{2}A + \frac{3}{4}B\right)|\psi\rangle = \frac{3}{2}\left[B - A\right]|\psi\rangle$$

For $|\psi\rangle = \left|j' = 1/2, j_{12} = 1, j_z' = \pm 1/2\right\rangle$ (the 2 states)

$$H|\psi\rangle = \left(2(A - B) + \frac{3}{4}B - \frac{3}{2}A + \frac{3}{4}B\right)|\psi\rangle = \frac{1}{2}\left[A - B\right]|\psi\rangle$$

When $j' = 3/2$, $j_{12} = 1$, and for (the 4 states) $|\psi\rangle = \left|j' = 3/2, j_{12} = 1, j_z' = \pm 3/2, \pm 1/2\right\rangle$,

$$H|\psi\rangle = \left(2(A - B) + \frac{15}{4}B - \frac{3}{2}A + \frac{3}{4}B\right)|\psi\rangle = \frac{1}{2}\left[A + B\right]|\psi\rangle$$

3 Coherent State

(3a) Normalization

$$|c\rangle_{\text{coh}} = e^{-|c|^2/2} \sum_n \frac{1}{n!} c^n (a^\dagger)^n |0\rangle = e^{-|c|^2/2} \sum_n \frac{1}{\sqrt{n!}} c^n |n\rangle$$

so that

$$\langle c|_{\text{coh}} |c\rangle_{\text{coh}} = e^{-|c|^2} \sum_n \frac{1}{n!} |c|^{2n} = 1$$

(3b) Orthogonality

$$\langle c_1|_{\text{coh}} |c_2\rangle_{\text{coh}} = e^{-(|c_1|^2+|c_2|^2)/2} \sum_n \frac{1}{n!} (\bar{c}_1 c_2)^n = e^{-(|c_1|^2+|c_2|^2)/2+\bar{c}_1 c_2}$$

To make this quantity vanish, the real part of $-(|c_1|^2+|c_2|^2)/2+\bar{c}_1 c_2$ must be infinite and negative, which is not possible for finite c_1 and c_2.

(3c) Eigenstate

$$a|c\rangle_{\text{coh}} = e^{-|c|^2/2} \sum_n \frac{1}{\sqrt{n!}} c^n a |n\rangle = e^{-|c|^2/2} \sum_n \frac{\sqrt{n}}{\sqrt{n!}} c^n |n-1\rangle = c|c\rangle_{\text{coh}}$$

(3d) Decomposition

We did this already in part (a),

$$|\langle n|c\rangle_{\text{coh}}|^2 = e^{-|c|^2} \frac{1}{n!} (|c|^2)^n$$

4 Sum Rule

(4a) Commutation

First, notice that $[x, V(x)] = 0$. Therefore,

$$[x, H] = \frac{1}{2m} \left([x, p]p + p[x, p] \right) = i\hbar \frac{p}{m}$$

so that

$$[[x, H], x] = i\hbar \frac{1}{m} [p, x] = i\hbar \frac{1}{m} (-i\hbar) = \frac{\hbar^2}{m}$$

(4b) The Sum Rule

Taking part (a) as a hint, we notice that

$$\frac{\hbar^2}{m} = [[x, H], x] = [xH - Hx, x] = xHx - Hx^2 - x^2H + xHx = 2xHx - Hx^2 - x^2H$$

or

$$xHx - \frac{Hx^2 - x^2H}{2} = \frac{\hbar^2}{2m}$$

Using this,

$$\begin{aligned}\sum_m (E_m - E_n)\,|\langle n|\,x\,|m\rangle|^2 &= \sum_m \langle n|\,x\,|m\rangle\,(E_m - E_n)\,\langle m|\,x\,|n\rangle = \langle n|\,x\left[\sum_m (E_m - E_n)\,|m\rangle\,\langle m|\right]x\,|n\rangle \\ &= \langle n|\,x\left[\sum_m (H - E_n)\,|m\rangle\,\langle m|\right]x\,|n\rangle = \langle n|\,x(H - E_n)\left[\sum_m |m\rangle\,\langle m|\right]x\,|n\rangle \\ &= \langle n|\,x(H - E_n)x\,|n\rangle = \langle n|\left(xHx - x^2E_n\right)|n\rangle = \langle n|\left(xHx - \frac{x^2E_n + E_nx^2}{2}\right)|n\rangle \\ &= \langle n|\left(xHx - \frac{x^2H + Hx^2}{2}\right)|n\rangle = \langle n|\,\frac{\hbar^2}{2m}\,|n\rangle = \frac{\hbar^2}{2m}\end{aligned}$$

(4c) Harmonic Oscillator

First, recall that $x = \sqrt{\frac{\hbar}{2m\omega}}\left(a + a^\dagger\right)$. It follows that

$$\langle n|\,x\,|m\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{m+1}\delta_{n,m+1} + \sqrt{m}\delta_{n+1,m}\right]$$

(4d) Harmonic Oscillator Sum Rule

$$\begin{aligned}\sum_n (E_n - E_m)\,|\langle n|\,x\,|m\rangle|^2 &= \hbar\omega\sum_n (n-m)\left|\sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{m+1}\delta_{n,m+1} + \sqrt{m}\delta_{n+1,m}\right]\right|^2 \\ &= \frac{\hbar^2}{2m}\sum_n (n-m)\left|\sqrt{n}\delta_{n,m+1} + \sqrt{m+1}\delta_{n+1,m}\right|^2 = \frac{\hbar^2}{2m}\left[(m+1) - m\right] = \frac{\hbar^2}{2m}\end{aligned}$$

5 Two Distinguishable 1/2 Spin Particles

(5a) Commutator

$$\left[\sigma_x^{(1)} \otimes \sigma_x^{(2)}, \sigma_y^{(1)} \otimes \sigma_y^{(2)}\right] = \sigma_x^{(1)}\sigma_y^{(1)} \otimes \sigma_x^{(2)}\sigma_y^{(2)} - \sigma_y^{(1)}\sigma_x^{(1)} \otimes \sigma_y^{(2)}\sigma_x^{(2)} = -\sigma_z^{(1)} \otimes \sigma_z^{(2)} + \sigma_z^{(1)} \otimes \sigma_z^{(2)} = 0$$

(5b) Spectrum

Just writing the terms down,

$$\frac{H}{h_0\hbar} = \begin{bmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{bmatrix} + \begin{bmatrix} 0 & -i\sigma_y \\ i\sigma_y & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & (-i)\cdot(-i) \\ 0 & 0 & (-i)\cdot i & 0 \\ 0 & i\cdot(-i) & 0 & 0 \\ i\cdot i & 0 & 0 & 0 \end{bmatrix} = 2\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So, we have two 0 energies ($|\pm\pm\rangle$), and $\pm 2h_0\hbar$ ($\frac{1}{\sqrt{2}}\left[|+-\rangle \pm |-+\rangle\right]$)

(5c)

This state is

$$|\psi(t=0)\rangle = |++\rangle$$

But, this is a zero energy eigenstate. So its time evolution is trivial (constant). We don't bother writing it down again.

(5d)

This state is also an eigenstate of $\sigma_z \otimes \sigma_z$, with eigenvalue $\hbar^2$, and hence this is also the expectation value.

Joe Bruin UID 000-000-0000

PHYSICS 221A

Final Exam – Fall 2013

Monday December 9th, 2013, at 3pm - 6pm

- Please write clearly, neat and logical presentation will be rewarded !
- Please write down your name and UID on the front page, if you separate sheets please write your name on each sheet.
- Make clear which question and which part you are answering on extra each page
- No core-dumps please !
- No books, notes, computers are allowed. You can use calculators but you won't need them ;-)
- Please turn off all electronic devices.
- All parts of questions, a),b)c) etc., carry equal weight unless otherwise indicated.

Good Luck !!

question	possible points	achieved points
1.	40	
2.	40	
3.	40	
4.	40	
5.	40	
Total	200	

Some possibly useful formulas

- The angular momentum algebra is given by $[J_1, J_2] = i\hbar J_3$, and cyclic permutations. The ladder operators, defined by $J_\pm \equiv J_1 \pm i J_2$, act as follows,

$$J_\pm |j,m\rangle = \hbar\sqrt{j(j+1) - m(m\pm 1)}\,|j, m\pm 1\rangle \tag{0.1}$$

where the states are properly normalized by $\langle j', m'|j, m\rangle = \delta_{j,j'}\delta_{m,m'}$.

- "little" CBH-formula

$$e^A B e^{-A} = \exp(Ad_A)B \tag{0.2}$$

where $Ad_X \cdot = [X, \cdot]$.

- "big" CBH-formula

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\frac{1}{12}[A,[A,B]]-\frac{1}{12}[B,[A,B]]+\cdots} \tag{0.3}$$

where the dots denote nested commutators of A, B of order 4 and higher.

- The Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{0.4}$$

- harmonic oscillator

$$\begin{aligned} a &= \frac{1}{\sqrt{2m\omega\hbar}}\Big(m\omega\, X + i\, P\Big) \\ a^\dagger &= \frac{1}{\sqrt{2m\omega\hbar}}\Big(m\omega\, X - i\, P\Big) \end{aligned} \tag{0.5}$$

- Position/momentum

$$[X, P] = i\hbar \tag{0.6}$$

QUESTION 1: Short questions [40 points]

a) If you add two spins j_1 and j_2 with eigenvalues of $\vec{J}_i^{\,2}$ being $\hbar^2 j_i(j_i+1)$, $i = 1, 2$ what are the possible values of j_{tot} for the total spin ? (state the result, no derivation necessary).

$$j_{tot} = j_1 + j_2,\ j_1 + j_2 - 1,\ \dots,\ |j_1 - j_2|$$

b) If an operator is both unitary and hermitian, what are its possible eigenvalues ? (give a brief argument)

$$U^+ U = 1 \quad \text{unitary}$$
$$U = U^+ \quad \text{hermitian}$$
$$\Rightarrow U^2 = 1 \Rightarrow \text{eigenvalues } \pm 1$$

c) You have three observables A, B, C they satisfy

$$[A, B] = 0, \qquad [A, C] = 0 \tag{0.7}$$

what can you say about $[A, [B, C]]$? (Justify your answer).

$$[A,[B,C]] + [B, \overset{0}{\cancel{[C,A]}}] + [C, \overset{0}{\cancel{[A,B]}}] = 0 \qquad \text{Jacobi identity}$$

$$\Rightarrow [A,[B,C]] = 0$$

d) State the Heisenberg uncertainty relation for general observables A, B and a general state $|\psi\rangle$.

$$\Delta_\psi^2 A \; \Delta_\psi^2 B \leq \frac{1}{4} |\langle\psi|[A,B]|\psi\rangle|^2$$

where

$$\Delta_\psi^2 \mathcal{O} = \langle\psi|(\mathcal{O} - \langle\mathcal{O}\rangle_\psi)^2|\psi\rangle$$

e) What condition (involving the Hamiltonian) must an observable to satisfy to be conserved ?

$$[\mathcal{O}, H] = 0$$

f) For a ladder operators of angular momentum show the following identities

$$J_+ J_- = (\vec{J})^2 - J_3^2 + \hbar J_3 \tag{0.8}$$

$$J_+ J_- = (J_1 + iJ_2)(J_1 - iJ_2)$$

$$= J_1^2 + J_2^2 + i(J_2 J_1 - J_1 J_2)$$

$$= J_1^2 + J_2^2 + J_3^2 - J_3^2 - i[J_1, J_2]$$

$$= \vec{J}^2 - J_3^2 + \hbar J_3$$

since $[J_1, J_2] = i\hbar J_3$

g) An $N \times N$ hermitian matrix P is an projector, i.e. $P^2 = P$, also the trace of P is $tr(P) = 3$. What is the dimension of the space the projector P projects on (i.e. what is the rank of the projection matrix)?

P can be diagonalized

$$P = \begin{pmatrix} 1_k & \\ & 0 \end{pmatrix} \qquad tr P = k \Rightarrow k = 3$$

Dimension of space is 3

h) for a three state system an observable and the state of the system are given by

$$A = a \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad |\,\psi\rangle = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \tag{0.9}$$

After you measure A and get $-a$ and a result, what is the new state of the system after the measurement ? (You do not need to normalize the state).

Eigen values $+a$ deg 2 $-a$ deg 1

Eigenstate of Eigenvalue $-a$: $\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

New state $\sim \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

- extra space -

QUESTION 2: [40 points] Consider a two-state system, say spin, with Hamiltonian

$$H_s = 2\omega S_x = \hbar\omega \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Here $S_{x,y,z} = \frac{\hbar}{2}\sigma_{x,y,z}$ where σ_i are the Pauli matrices. At time $t = 0$, the system is in the eigenstate of S_z with eigenvalue $+\frac{1}{2}\hbar$.

a) If you were to measure S_y at time t=0, what values would you get and what are the associated probabilities ?

b) (The rest of the problem is independent of a), i.e. no measurement has taken place). Compute the evolution operator $U(t)$ for this system.

c) Evaluate the probability as a function of time t, called $P(t)$, for the system to be measured at time t in the eigenstate of S_z with eigenvalue $-\frac{1}{2}\hbar$.

d) Find the form of the operator $S_y(t)$ in the Heisenberg picture (assuming at $t = 0$ Heisenberg and Schrödinger picture agree).

a) state at $t=0$ $\quad |\psi\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ eigenstates $|S_y,+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$S_y -\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\Rightarrow |\psi\rangle = \frac{1}{\sqrt{2}}|S_y +\rangle + \frac{1}{\sqrt{2}}|S_y, -\rangle$

$\Rightarrow S_y = +\hbar/2$ with $P = \frac{1}{2}$

$S_y = -\frac{\hbar}{2}$ with $P = -1/2$

- extra space -

b) $\mathcal{U}(t) = \exp\left(-\frac{i}{\hbar} t H\right)$

$$= \exp\left(-i\omega t \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$$

$$= \begin{pmatrix} \cos\omega t & -i\sin\omega t \\ -i\sin\omega t & \cos\omega t \end{pmatrix}$$

c) $|\psi(t)\rangle = \mathcal{U}(t)\, |\psi(t=0)\rangle$

$$= \begin{pmatrix} \cos\omega t & -i\sin\omega t \\ -i\sin\omega t & \cos\omega t \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\omega t \\ -i\sin\omega t \end{pmatrix} = \cos\omega t \begin{pmatrix} 1 \\ 0 \end{pmatrix} - i\sin\omega t \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$P(t) = |\sin\omega t|^2$

- extra space -

d) $|\psi\rangle_H = U^{-1}(t)\, |\psi(t)\rangle_S$

$\mathcal{O}_H = U^{-1} \mathcal{O}_S\, U$

$\Rightarrow \mathcal{O}_S = S_y = \frac{\hbar}{2} \sigma_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$$\mathcal{O}_H = \frac{\hbar}{2} \begin{pmatrix} \cos\omega t & i\sin\omega t \\ i\sin\omega t & \cos\omega t \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \cos\omega t & -i\sin\omega t \\ -i\sin\omega t & \cos\omega t \end{pmatrix}$$

$$= \frac{\hbar}{2} \begin{pmatrix} \cos\omega t & i\sin\omega t \\ i\sin\omega t & \cos\omega t \end{pmatrix} \begin{pmatrix} -\sin\omega t & -i\cos\omega t \\ i\cos\omega t & \sin\omega t \end{pmatrix}$$

$$= \frac{\hbar}{2} \begin{pmatrix} -2\sin\omega t\cos\omega t & -i(\cos^2\omega t - \sin^2\omega t) \\ i(\cos^2\omega t - \sin^2\omega t) & 2\sin\omega t\cos\omega t \end{pmatrix}$$

- extra space -

QUESTION 3: [40 points]

Let $a_\pm$ be two independent harmonic oscillator lowering operators, satisfying

$$[a_+, a_-] = [a_+, a_-^\dagger] = [a_+^\dagger, a_-] = [a_+^\dagger, a_-^\dagger] = 0$$
$$[a_+, a_+^\dagger] = [a_-, a_-^\dagger] = 1$$

We define the operators

$$N = \frac{1}{2}(a_+^\dagger a_+ + a_-^\dagger a_-) \qquad J_z = \frac{\hbar}{2}(a_+^\dagger a_+ - a_-^\dagger a_-) \qquad J_\pm = \hbar a_\pm^\dagger a_\mp$$

a) Show that $J_z, J_\pm$ satisfy the angular momentum commutation relations, i.e.

$$[J_+, J_-] = 2\hbar J_z, \quad [J_z, J_\pm] = \pm\hbar J_\pm \tag{0.10}$$

b) Show that

$$[\vec{J}^2, J_z] = 0$$

c) Show that the spin $\frac{1}{2}$ quantum system is the eigenspace of N with eigenvalue $1/2$.

d) Express the vectors in the the Hilbert space of the two harmonic oscillators with $N \leq 1$ in terms of eigenstates of J_z, $\vec{J}^2$. .

Hint: For part c) and d) you can use without proof the identity: $\vec{J}^2 = \hbar^2 N(N+1)$.

a) $[J_z, J_+] = \frac{\hbar^2}{2}[a_+^\dagger a_+ - a_-^\dagger a_-, a_+^\dagger a_-]$

$= \frac{\hbar^2}{2}\left(a_+^\dagger [a_+, a_+^\dagger] a_- - a_+^\dagger [a_-^\dagger, a_-] a_- \right)$

$= \frac{\hbar^2}{2}\left(a_+^\dagger a_- + a_+^\dagger a_-\right) = \hbar J_+$

$[J_z, J_-] = \frac{\hbar^2}{2}[a_+^\dagger a_+ - a_-^\dagger a_-, a_-^\dagger a_+]$

$= \frac{\hbar^2}{2}[a_-^\dagger [a_+^\dagger, a_+] a_+ - a_-^\dagger [a_-, a_-^\dagger] a_+)$

$= -\frac{\hbar^2}{2}\left(a_-^\dagger a_+ + a_-^\dagger a_+\right) = -\hbar J_-$

- extra space -

$$[J_+, J_-] = \hbar^2 [a_+^+ a_-, a_-^+ a_+]$$

$$= \hbar^2 \left(a_+ [a_-, a_-^+] a_+ + a_-^+ [a_+^+, a_+] a_-\right]$$

$$= \hbar^2 \left(a_+^+ a_+ - a_-^+ a_-\right) = 2\hbar J_z$$

b) $[\vec{J}^2, J_z] = 0$

$$\vec{J}^2 = J_z^2 + J_+ J_- - \hbar J_z$$

$$[\vec{J}^2, J_z] = [J_+ J_-, J_z]$$

$$= J_+ [J_-, J_z] + [J_+, J_z] J_-$$

$$= -\hbar J_+ J_- + \hbar J_+ J_- = 0$$

c) $|n, m; N\rangle = \frac{1}{\sqrt{n!}} (a_+^+)^n (a_-^+)^m |0\rangle$

$|1,0,N\rangle$ & $|0,1,N\rangle$ have

$N = 1/2 \Rightarrow \vec{J}^2$ ev. $\hbar^2 \frac{3}{4} \Rightarrow j = 1/2$

$J_z \, |1,0,N\rangle = +\frac{\hbar}{2}$

$J_z \, |0,1,N\rangle = -\frac{\hbar}{2}$

$\Rightarrow \quad |1,0\rangle = |j=1/2 \; m=1/2\rangle$

$|0,1\rangle = |j=1/2, m=-1/2\rangle$

also check that $|1,0\rangle$ & $|0,1\rangle$ are properly normalized

d) $\underline{N = 1}$ states:

	J_z	
$\frac{1}{\sqrt{2}}(a_+^\dagger)^2 \|0\rangle$	$\hbar$	$\|j=1, m=1$
$\frac{1}{\sqrt{2}}(a_-^\dagger)^2 \|0\rangle$	$-\hbar$	$\|j=1, m=-1$
$a_+^\dagger a_-^\dagger \|0\rangle$	0	$\|j=0, m=0\rangle$

Check normalization

- extra space -

QUESTION 4: [40 points]

A coherent state for a single harmonic oscillator is given by

$$\mid c\rangle_{\text{coh}} = C e^{ca^\dagger} \mid 0\rangle \tag{0.11}$$

Where $\mid 0\rangle$ is the ground state of the harmonic oscillator and C is a normalization constant.

a) Show that $\mid c\rangle_{\text{coh}}$ is normalized to 1 for $C = e^{-|c|^2/2}$.

b) Calculate $a \mid c\rangle_{\text{coh}}$ and the expectation value ${}_{\text{coh}}\langle c \mid N \mid c\rangle_{\text{coh}}$, where N is the number operator of the harmonic oscillator.

c) Calculate the expectation value of the position operator x and momentum operator p, for the state $\mid c\rangle_{\text{coh}}$

d) Consider the system which at time $t = 0$ is in the state $\mid c_0\rangle$, calculate the expectation value of x at time t: $\langle x\rangle_t$ and the expectation value of p at time t: $\langle p\rangle_t$ and show that they satisfy

$$\langle p\rangle_t = m\frac{d}{dt}\langle x\rangle_t \tag{0.12}$$

a) $_{coh}\langle c|c\rangle_{coh} = |C|^2 \langle 0 | e^{c^* a} e^{c a^+} |0\rangle$

Use BCH for $[A,B] = \text{const}$

$e^A e^B = e^{A+B} e^{\frac{1}{2}[A,B]}$

$e^B e^A = e^{A+B} e^{-\frac{1}{2}[A,B]}$

$\Rightarrow e^A e^B = e^B e^A e^{[A,B]}$

$\langle c|c\rangle = |C|^2 \langle 0|0\rangle \cdot e^{|c|^2 [a,a^+]}$, 1

normalized if $|C|^2 = e^{-|c|^2}$ or $C = e^{-|c|^2/2}$

- extra space -

b) $$a\,|c\rangle_{coh} = e^{-|c|^2/2}\, a\, e^{ca^+}\,|0\rangle$$

Use little BCH.

$$e^{-ca^+}\, a\, e^{ca^+} = \exp(-\mathrm{Ad}_{ca^+})\, a$$

$$= a - c\,[a^+, a]$$

$$= a + c$$

$$a\, e^{ca^+} = e^{ca^+}(a+c)$$

$$a\,|c\rangle_{coh} = +c\,|c_{coh}\rangle$$

$${}_{coh}\langle c|\, a^+ = +\, c^*\, {}_{coh}\langle c|$$

$${}_{coh}\langle c|\, N\, |c_{coh}\rangle = \langle c|\, a^+ a\, |c\rangle$$

$$= |c|^2$$

d) Either use Heisenberg picture or Schrödinger picture

in Schrödinger picture.

$$|\psi(t)\rangle = \exp(-iHt)|c_0\rangle$$

$$= C\sum_n \frac{c_0^n}{\sqrt{n!}} e^{-iHt}(a^\dagger)^n|0\rangle$$

$$= C\sum_n \frac{c_0^n}{\sqrt{n!}} e^{-i(n+1/2)\omega t}(a^\dagger)^n|0\rangle$$

$$= C\, e^{-i\frac{\omega t}{2}} \sum_n \frac{c(t)^n}{\sqrt{n!}}(a^\dagger)^n|0\rangle$$

unimportant phase

$$= e^{-\frac{i\omega t}{2}}\, C\, |c(t)\rangle$$

$$c(t) = c_0\, e^{-i\omega t} = |c_0|\, e^{-i(\omega t+\delta)}$$

$$\langle\psi(t)|x|\psi\rangle = \sqrt{\frac{2\hbar}{m\omega}}\,\mathrm{Re}\, c(t) = \sqrt{\frac{2\hbar}{m\omega}}\,|c_0|(\cos\omega t+\delta)$$

$$\langle\psi(t)|p|\psi(t)\rangle = \sqrt{2m\hbar\omega}\,\mathrm{Im}\, c(t) = -\sqrt{2m\hbar\omega}\,|c_0|\sin(\omega t+\delta)$$

by inspection $\quad \langle\dot{x}\rangle = \frac{\langle p\rangle}{m}.$

- extra space -

c) $\langle 0| e^{c^* a} (a^+)$

BCH again $e^{c^* a} a^+ e^{-c^* a} = a^+ + c^* [a, a^+]$

$= a^+ + c^*$

$\Rightarrow \langle 0| e^{c^* a} a^+ = c^* \langle 0| e^{c^* a}$

$\langle c| \hat{x} |c\rangle = \frac{\sqrt{\hbar}}{\sqrt{2m\omega}} \langle c| a + a^+ |c\rangle$

$= \frac{\sqrt{\hbar}}{\sqrt{2m\omega}} (c + c^*)$

$= \sqrt{\frac{2\hbar}{m\omega}} \operatorname{Re} c$

$\langle c| \hat{p} |c\rangle = i \sqrt{\frac{m\hbar\omega}{2}} \langle c|(a^+ - a)|c\rangle$

$= i \sqrt{\frac{m\hbar\omega}{2}} (c^* - c)$

$= \sqrt{2m\hbar\omega} \operatorname{Im} c$

QUESTION 5: [40 points]

Consider a system of two spin 1/2 particles, labelled a and b, with respective spin operators $\vec{S}_a$ and $\vec{S}_b$. We ignore all quantum numbers but those of spin. The eigenstates of $(\vec{S}_a)^2$ and $(S_a)_z$ are labelled my $\mid j_a, m_a\rangle$ and he eigenstates of $(\vec{S}_b)^2$ and $(S_b)_z$ are labelled my $\mid j_b, m_b\rangle$

a) For the total spin $\vec{S}_{tot} = \vec{S}_a + \vec{S}_b$, what are the possible eigenvalues of the $(\vec{S}_{tot})^2$ and $(S_{tot})_z$ which can appear (They are labelled by the numbers j, m and call the associated eigenstates $\mid j, m\rangle$).

b) Find the expression for the states $\mid j, m\rangle$ in terms of $\mid m_a, m_b\rangle$.

c) Find the expectation value for $(S_a)_z$ for the states $\mid j, m\rangle$

d) The particles are in the state $\mid \psi\rangle$ of **zero total angular momentum**, which we consider normalized to $\langle\psi|\psi\rangle = 1$.
Let $\hat{n}_a$ and $\hat{n}_b$ be two independent unit vectors. Compute the expectation value of the product of the spin operators projected onto the directions $\hat{n}_a$ and $\hat{n}_b$ respectively, namely,

$$\langle\psi \mid (\vec{n}_a \cdot \vec{S}_a)(\vec{n}_b \cdot \vec{S}_b) \mid \psi\rangle \tag{0.13}$$

a) $j_1 = 1/2 \quad j_2 = 1/2$

$j_{tot} = 0, 1$

$j_{tot} = 1 \quad j_z = \pm 1, 0$

$j_{tot} = 0 \quad j_z = 0$

- extra space -

b) $|j=1, m=1\rangle = |m_1 = 1/2\ m_2 = 1/2\rangle$

$|j=1, m=0\rangle = \frac{1}{\sqrt{2}}\left(|m=1/2, m_2=-1/2\rangle + |m=-1/2, m_2=+1/2\rangle\right)$

$|j=1, m=-1\rangle = |m_1=-1/2, m_2=-1/2\rangle$

$|j=0, m=0\rangle = \frac{1}{\sqrt{2}}\left(|m_1=1/2, m_2=-1/2\rangle - |m_1=-1/2, m_2=+1/2\rangle\right)$

c) $\langle j=1, m=1 | (S_a)_z | j=1, m=1\rangle = \frac{\hbar}{2}$

$\langle j=1, m=-1 | S_{az} | j=1, m=-1\rangle = -\frac{\hbar}{2}$

$\langle j=1, m=0 | S_{az} | j=1, m=0\rangle = \frac{1}{\sqrt{2}}\left(\hbar/2 - \frac{\hbar}{2}\right) = 0$

$\langle j=0, m=0 | S_{az} | j=0, m=0\rangle = \frac{1}{\sqrt{2}}\left(\hbar/2 - \hbar/2\right) = 0$

- extra space -

d) $|\psi\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle_a|-\rangle_b - |-\rangle_a|-\rangle_b\right)$

from above.

Result: linear in $\hat{n}_a$ & $\hat{n}_b$ and rotationally invariant

Result must be of the form:

$$C\,\hat{n}_a\cdot\hat{n}_b$$

where C is independent of $\hat{n}_a$ & $\hat{n}_b$

Choose $\hat{n}_a = \begin{pmatrix}0\\0\\1\end{pmatrix}$ & $\hat{n}_b = \begin{pmatrix}0\\0\\1\end{pmatrix}$

$$S_{az}S_{bz}|\psi\rangle = -\frac{\hbar^2}{2}|\psi\rangle$$

$$\Rightarrow C = -\frac{\hbar^2}{2}$$

- extra space -

Name: UID:

PHYSICS 221B

Practice Midterm Exam – Winter 2014

Real Exam: Monday Feb 3rd in class

- Please write clearly
- The order of problems is not by difficulty.
- Print your name on every page used, including this one;
- Make clear which question and which part you are answering on each page.
- No core-dumps please !
- No books, notes, computers, or calculators are allowed during the exam;
- Please turn off all electronic devices.

Good Luck !!

question	possible points	achieved points
1.	30	
2.	40	
3.	40	
Total	110	

Some possibly useful formulas

- Harmonic oscillator for a Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 = \hbar\omega(a^\dagger a + \frac{1}{2}) \tag{0.1}$$

with

$$a = \frac{1}{\sqrt{2m\hbar\omega}}(m\omega x + ip) \qquad\qquad [a, a^\dagger] = 1 \tag{0.2}$$

- The angular momentum algebra is $[J^1, J^2] = i\hbar J^3$, and two cyclic permutations thereof. The corresponding ladder operators are defined to be $J^\pm = J^1 \pm iJ^2$, and act by

$$J^\pm|j,m\rangle = \hbar\sqrt{j(j+1) - m(m\pm 1)}\,|j, m\pm 1\rangle \tag{0.3}$$

- Spherial harmonics

$$Y_0^0 = \frac{1}{\sqrt{4\pi}} \tag{0.4}$$

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos theta \tag{0.5}$$

$$Y_1^{\pm 1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\phi} \tag{0.6}$$

QUESTION 1: [30pts]

Consider a pair of harmonic oscillators with unperturbed Hamiltonian

$$\hbar\omega_1(a_1^\dagger a_1 + \frac{1}{2}) + \hbar\omega_2(a_2^\dagger a_2 + \frac{1}{2}) \tag{0.7}$$

where $\omega_1 < \omega_2 < 2\omega_1$. A perturbation is added

$$H_1 = \epsilon(a_1^\dagger a_1^\dagger a_2 + a_2^\dagger a_1 a_1) \tag{0.8}$$

a) Find the three lowest eigenstates of the **unperturbed** Hamiltonian.

b) Show that two of the three lowest levels are exact eigenstates of $H_0 + H_1$.

c) Calculate the first non vanishing correction to the energy and the state for the third.

QUESTION 2: [40 points]

A relativistic particle in one dimension with mass m is subject to a harmonic oscillator potential, and governed by the following Hamiltonian,

$$H_c = \sqrt{m^2c^4 + p^2c^2} - mc^2 + \frac{1}{2}m\omega^2 x^2 \tag{0.9}$$

where $[x, p] = i\hbar$.

a) Show that n the limit $c \to \infty$, the Hamiltonian H_c reduces to the standard non-relativistic harmonic oscillator Hamiltonian (which will be denoted here by H_∞).

b) Using perturbation theory in power of $1/c^2$, compute the leading relativistic correction to the ground state energy of H_∞.

c) For the general case of finite c show that, in a basis where p is **diagonal**, the spectrum of H_c may be obtained by solving a Schrödinger-like differential equation.

d) Estimate the ground state energy using the variational method for the equation in c).

QUESTION 3: [40 points]

We consider the Hamiltonian of a rigid rotator given by

$$H_0 = \frac{\vec{L}^2}{2I} \tag{0.10}$$

Where I is the moment of inertia.
It is assumed that the rigid rotator has a magnetic moment and is placed in an uniform electric field in the z-direction. Averaging over the radial dependence this amounts to adding a perturbation

$$H' = -\epsilon\mu\cos\theta \tag{0.11}$$

(i.e. For the problem you can neglect any radial dependence and treat the problem as one which only depends on the angular coordinates θ and ϕ).

a) Find the spectrum and degeneracies of H_0.

b) Using the following relation

$$\cos\theta\, Y_l^m = \sqrt{\frac{(l+1)^2 - m^2}{(2l+1)(2l+3)}} Y_{l+1}^m + \sqrt{\frac{l^2 - m^2}{(2l+1)(2l-1)}} Y_{l-1}^m \tag{0.12}$$

where Y_l^m are the spherical harmonics in standard spherical coordinatesl, Calculate the following matrix elements

$$\langle lm|\cos\theta|l'm'\rangle \tag{0.13}$$

Hint: very few of the matrix elements are nonzero.

c) Using the results of b) argue that: First, the first order contribution in perturbation theory to the energy of the state $\mid lm\rangle$ vanishes. Second, even though the spectrum is degenerate one can apply second order perturbation theory

d) Calculate the second oder contribution to the shift in the energy for the state $\mid lm\rangle$.

Quantum Mechanics: 221B, Practice Midterm

Antonio Russo

Real Exam: February 3, 2014

1 Two SHOs

$$H = \hbar\omega_1(N_1 + 1/2) + \hbar\omega_2(N_2 + 1/2) \quad \text{and} \quad H_1 = \epsilon(a_1^\dagger a_1^\dagger a_2 + a_2^\dagger a_1 a_1)$$

(a) Unperturbed States

The lowest eigenstates of H are

$$|00\rangle \quad \text{with} \quad E_{00} = \frac{\hbar}{2}\left[\omega_1 + \omega_2\right)$$

and

$$|10\rangle \quad \text{with} \quad E_{10} = \frac{\hbar}{2}\left[3\omega_1 + \omega_2\right)$$

and

$$|01\rangle \quad \text{with} \quad E_{01} = \frac{\hbar}{2}\left[\omega_1 + 3\omega_2\right)$$

(b) Lowest Levels

Notice that

$$H_1 |00\rangle = \epsilon(a_1^\dagger a_1^\dagger a_2 + a_2^\dagger a_1 a_1) |00\rangle = 0$$

and

$$H_1 |10\rangle = \epsilon(a_1^\dagger a_1^\dagger a_2 + a_2^\dagger a_1 a_1) |10\rangle = 0$$

(so these states are unaffected).

(c) Lowest correction for $|10\rangle$

Notice that

$$H_1 |01\rangle = \epsilon(a_1^\dagger a_1^\dagger a_2 + a_2^\dagger a_1 a_1) |01\rangle = \epsilon |20\rangle$$

It follows that the first order correction to the energy level is zero. The second order correction is (noticing there is no degeneracy)

$$\Delta E_{01}^{(2)} = \epsilon^2 \frac{1}{E_{01}^{(0)} - E_{20}^{(0)}} = \frac{\epsilon^2}{\hbar\left[\omega_2 - 2\omega_1\right]}$$

2 Relativistic SHO

(a) H_∞

$$H_c = mc^2\left[\sqrt{1 + \frac{p^2}{m^2c^2}} - 1\right] + \frac{1}{2}m\omega^2x^2 = mc^2\left[1 + \frac{1}{2}\frac{p^2}{m^2c^2} - \frac{1}{8}\frac{p^4}{m^4c^4} + o\left(\frac{p^2}{m^2c^2}\right)^3 - 1\right] + \frac{1}{2}m\omega^2x^2$$
$$\doteq \frac{p^2}{2m} - \frac{1}{8}\frac{p^4}{m^3c^2} + \frac{1}{2}m\omega^2x^2$$

As $c \to \infty$, we just get the SHO Hamiltonian.

(b) Leading Correction

Notice that

$$p = -i\sqrt{\frac{2}{m\hbar\omega}}$$

so that

$$H_0 = \hbar\omega(a^\dagger a + 1/2) \quad \text{and} \quad H_1 - \frac{1}{8}\frac{p^4}{m^3c^2} = -\frac{\hbar^2\omega^2}{32mc^2}\left(a - a^\dagger\right)^4$$

(notice that there is no degeneracy). The first order energy shift is

$$\Delta E^{(1)} = \langle 0|\, H_1 \,|0\rangle = -\frac{\hbar^2\omega^2}{32mc^2}\,\langle 0|\,(a - a^\dagger)^4\,|0\rangle$$

The final product has $2^4 = 16$ terms, but vanishes unless the leftmost and rightmost terms are a and $a^\dagger$, respectively. Furthermore, the number of a and $a^\dagger$ must be equal. This leaves us with just two terms:

$$\begin{aligned}\langle 0|\,(a - a^\dagger)^4\,|0\rangle &= \langle 0|\, a\left[aa^\dagger + a^\dagger a\right] a^\dagger\,|0\rangle = \langle 0|\, a\left[1 + 2a^\dagger a\right] a^\dagger\,|0\rangle \\ &= \langle 0|\left[aa^\dagger + 2aa^\dagger aa^\dagger\right]|0\rangle = \langle 0|\left[3aa^\dagger\right]|0\rangle = 3\end{aligned}$$

so that

$$\Delta E^{(1)} = -\frac{3\hbar^2\omega^2}{32mc^2}$$

(c) Momentum Basis

Put

$$\psi(p) = \langle p|\psi\rangle$$

so that

$$|\psi\rangle = I\,|\psi\rangle = \int \frac{1}{2\pi}\,|p\rangle\,\langle p|\psi\rangle\; dp = \frac{1}{2\pi}\int \psi(p)\,|p\rangle\; dp$$

Then,

$$\hat{p}\,|\psi\rangle = \frac{1}{2\pi}\int p\psi(p)\,|p\rangle\; dp$$

and

$$\hat{x}\,|\psi\rangle = \frac{1}{2\pi}\int (i\hbar)\psi'(p)\,|p\rangle\; dp$$

Thus,

$$H_c\,|\psi\rangle = \frac{1}{2\pi}\int\left[\sqrt{m^2c^4 + p^2c^2} - mc^2 - \frac{1}{2}m\omega^2(\hbar\partial_p)^2\right]\psi(p)\,|p\rangle\; dp$$

Putting $\mu = \frac{1}{m\omega^2}$ and $V(p) = \sqrt{m^2c^4 + p^2c^2} - mc^2$,

$$H_c\,|\psi\rangle = \frac{1}{2\pi}\int\left[-\frac{\hbar^2}{2\mu}\partial_p^2 + V(p)\right]\psi(p)\,|p\rangle\; dp$$

which is "Schrödinger-like".

3 Disturbed Rigid Rotator

$$H_0 = \frac{\boldsymbol{L}^2}{2I} \quad \text{and} \quad H' = -\epsilon\mu\cos\theta$$

(a) H_0

We recall the eigenvalues of L^2:

$$H_0\,|l, m\rangle = \frac{1}{2I}\cdot\hbar^2 l(l+1)\,|l, m\rangle$$

where $l \in \mathbb{Z}$ with $l \geq 0$, and $m \in \mathbb{Z}$ with $|m| \leq l$

(b) The Math

$$\langle lm|\cos\theta\,|l'm'\rangle = \int d\Omega (Y_l^m)^* \cos\theta Y_{l'}^{m'} = \int d\Omega (Y_l^m)^* \left[\sqrt{\frac{(l+1)^2-m^2}{(2l+1)(2l+3)}}Y_{l'}^{m'+1} + \sqrt{\frac{l^2-m^2}{(2l+1)(2l-1)}}Y_{l'}^{m'-1}\right]$$
$$= \delta_{mm'}\left[\delta_{l,l'+1}\overbrace{\sqrt{\frac{(l+1)^2-m^2}{(2l+1)(2l+3)}}}^{\alpha_{l,+}} + \delta_{l,l'-1}\overbrace{\sqrt{\frac{l^2-m^2}{(2l+1)(2l-1)}}}^{\alpha_{l,-}}\right]$$

(c) The Reasoning

Notice that H' does not allow mixing of different m values; i.e., $\langle lm|\,H'\,|l'm'\rangle \propto \delta_{mm'}$. We can therefore treat each m sector separately. In any m sector, it is clear that there is no longer any degeneracy, and the first order corrections vanish (since there are no diagonal terms).

(d) Second Order

The second order correction is

$$E_{lm}^{(2)} = -\sum_{l'} \frac{|\langle l'm'|\,H'\,|lm\rangle|^2}{E_{lm}^{(0)} - E_{l'm'}^{(0)}} = -\frac{2I}{\hbar^2}\mu^2 \sum_{l'} \frac{\delta_{l,l'+1}\alpha_{l,+}^2 + \delta_{l,l'-1}\alpha_{l,-}^2}{l'(l'+1) - l(l+1)}$$
$$= -\frac{2I}{\hbar^2}\mu^2 \left[\Theta(l-1)\frac{\alpha_{l,+}^2}{l'(l'-1) - l(l+1)} + \frac{\alpha_{l,-}^2}{(l'+2)(l'+1) - l(l+1)}\right]$$

The Θ just means that this term doesn't appear for $l = 0$.

Name: UID:

PHYSICS 221B

Midterm Exam – Winter 2014

- Please write clearly
- The order of problems is not by difficulty.
- Print your name and UID on the front page. If you separate pages, please write you name on all of them.
- Make clear which question and which part you are answering on each page.
- No core-dumps please !
- No books, notes, computers, or calculators are allowed during the exam;
- Please turn off all electronic devices.

Good Luck !!

question	possible points	achieved points
1.	30	
2.	40	
3.	50	
Total	120	

Some possibly useful formulas

- Harmonic oscillator for a Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 = \hbar\omega(a^\dagger a + \frac{1}{2}) \tag{0.1}$$

with

$$a = \frac{1}{\sqrt{2m\hbar\omega}}(m\omega x + ip)\,, \quad a^\dagger = \frac{1}{\sqrt{2m\hbar\omega}}(m\omega x - ip)\,, \quad [a, a^\dagger] = 1 \tag{0.2}$$

- The angular momentum algebra is $[J^1, J^2] = i\hbar J^3$, and two cyclic permutations thereof. The corresponding ladder operators are defined to be $J^\pm = J^1 \pm iJ^2$, and act by

$$J^\pm|j,m\rangle = \hbar\sqrt{j(j+1) - m(m\pm 1)}\,|j, m\pm 1\rangle \tag{0.3}$$

- The Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{0.4}$$

QUESTION 1: [30pts] Neutral Kaons

The Neutral Kaon K^0 and the neutral anti-Kaon $\bar{K}^0$ are degenerate with respect to the QCD hamiltonian H_0

$$H_0 \mid K_0\rangle = E_K \mid K_0\rangle, \qquad H_0 \mid \bar{K}_0\rangle = E_K \mid \bar{K}_0\rangle, \tag{0.5}$$

Where $E_K = 500\ MeV$. We define two more discrete operations. First (intrinsic) parity is a unitary operator P which acts as follows:

$$P \mid K_0\rangle = - \mid K_0\rangle, \qquad P \mid \bar{K}_0\rangle = - \mid \bar{K}_0\rangle \tag{0.6}$$

and charge conjugation C is a unitary operator and acts as follows:

$$C \mid K_0\rangle = \mid \bar{K}_0\rangle, \qquad C \mid \bar{K}_0\rangle = \mid K_0\rangle \tag{0.7}$$

a) Find the eigenvalues and eigenstates of the operator CP in the two dimensional state space spanned by $\mid K_0\rangle$ and $\mid \bar{K}_0\rangle$.

b) Assume that a (weak) interaction H_1 is added to H_0 which commutes with CP. Find the most general form of H_1 (viewed as a 2x2 matrix in the state space spanned by $\mid K_0\rangle$ and $\mid \bar{K}_0\rangle$.

c) Calculate the first order shift of the energies due to perturbation H_1.

In the basis $\begin{pmatrix} |K_0\rangle \\ |\bar{K}_0\rangle \end{pmatrix}$ the operators H_0, P, C look as follows

$$H_0 = \begin{pmatrix} E_K & 0 \\ 0 & E_K \end{pmatrix} \qquad P = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \qquad C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

a) $PC = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ $\qquad \det(1\lambda - PC) = 0 \qquad \lambda^2 = 1 \Rightarrow \lambda = \pm 1$

eigenstates:
$$|+\rangle = \frac{1}{\sqrt{2}}\left(|K_0\rangle + |\bar{K}_0\rangle\right)$$
$$|-\rangle = \frac{1}{\sqrt{2}}\left(|K_0\rangle - |\bar{K}_0\rangle\right)$$

- extra space - Please state which problem you are working on !

b) $PC = -\sigma_1$ hence in the basis of 2x2 matrices $1, \sigma_i$ PC

Comutes with $H_1 = a\mathbb{1} + b\cdot\sigma_1 = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$

c) Degenerate perturbation theory.

$H_0 + \lambda H_1$ $\qquad$ $E_i^{(1)}$ are eigenvalues of H_1

$$\det(\mathbb{1}E^{(1)} - H) = \det\begin{pmatrix} a-E & b \\ b & a-E \end{pmatrix}$$

$$= (a-E)^2 - b^2$$

$$E_1^{(1)} = a - b$$

$$E_2^{(1)} = a + b$$

QUESTION 2: [50 points]

Consider the Hamiltonian for a rigid rotator

$$H = \frac{L_1^2}{2I_1} + \frac{L_2^2}{2I_2} + \frac{L_3^2}{2I_3} \tag{0.8}$$

Here L_i are the angular momentum operators and $I_i, i = 1, 2, 3$ are constants denoting the moments of inertia around the three axis.

a) For the case of the symmetric top, $I = I_1 = I_2$ and $I_3 \neq I_1$ and one has

$$H_0 = \frac{L_1^2}{2I} + \frac{L_2^2}{2I} + \frac{L_3^2}{2I_3} \tag{0.9}$$

derive the energy levels and their degeneracies.

b) With the definitions $I = \frac{1}{2}(I_1 + I_2)$ and $\Delta = \frac{1}{2}(I_1 - I_2)$. Express the Hamiltonian for a slightly non-symmetric top, i.e. $|\Delta| << I$ and $|\Delta| << |I - I_3|$ as

$$H = H_0 + \Delta \times H_1 + o(\Delta^2) \tag{0.10}$$

and determine H_1

c) Calculate the correction energy eigenvalue of the state with $l = 0$ to first order in Δ.

d) Calculate the correction energy eigenvalue of the states with $l = 1$ to first order in Δ. State carefully which kind of perturbation theory (degenerate or non-degenerate) you use for which states.

a) $H_0 = \frac{1}{2I}(L_1^2 + L_2^2) + \frac{1}{2I_3} L_3^2$

$= \frac{1}{2I}(L_1^2 + L_2^2 + L_3^2) + \left(\frac{1}{2I_3} - \frac{1}{2I}\right) L_3^2$

$|l, m\rangle$ eigenstate of $\vec{L}^2$ & L_3

- extra space - Please state which problem you are working on !

$$H_0 |\ell, m\rangle = \frac{\hbar^2}{2I}\ell(\ell+1) + \frac{I - I_3}{2I_3 I}\hbar^2 m^2$$

For generic values of I & I_3

$m \neq 0$ 2 fold degeneracy m & $-m$

$m = 0$ no degeneracy

b) non symmetric $I = \frac{1}{2}(I_1 + I_2)$ $\Delta = \frac{1}{2}(I_1 - I_2)$

$\rightarrow I_1 = I + \Delta$ $I_2 = I - \Delta$

$$H = \frac{L_1^2}{2(I+\Delta)} + \frac{L_2^2}{2(I-\Delta)} + \frac{L_3^2}{2I_3}$$

$$= \underbrace{\frac{L_1^2}{2I} + \frac{L_2^2}{2I} + \frac{L_3^2}{2I_3}}_{H_0} + \underbrace{\left(-\frac{L_1^2}{2I^2} + \frac{L_2^2}{2I^2}\right)}_{H_1}\Delta + o(\Delta)$$

$$\boxed{H_1 = \frac{1}{2I^2}\left(L_2^2 - L_1^2\right)}$$

- extra space - Please state which problem you are working on !

c) $\ell = 0 \quad m = 0$ unique state

non degenerate perturbation theory.

$L_i |\ell=0, m=0\rangle = 0$

$$\Rightarrow \quad E^{(1)} = \langle 0,0 | H_1 | 0,0 \rangle$$
$$= 0$$

no condition

d) $\ell = 1 \quad m = \pm 1$ degenerate

$m = 0$ non degenerate

$$E^{(1)}_{\ell=1, m=0} = \langle \ell=1, m=0 | H_1 | \ell=1, m=0 \rangle$$
$$= \frac{\Delta}{2I} \langle \ell=1, m=0 | L_2^2 - L_1^2 | \ell=1, m=0 \rangle$$

$$(L_+)^2 = (L_1 + i L_2)^2 = L_1^2 + i(L_1 L_2 + L_2 L_1) - L_2^2$$
$$(L_-)^2 = (L_1 - i L_2)^2 = L_1^2 - i(L_1 L_2 + L_2 L_1) - L_2^2$$
$$\Rightarrow L_2^2 - L_1^2 = -(L_+^2 + L_-^2)$$

Since $L_\pm^2 |\ell=1, m=0\rangle = 0 \quad \boxed{E^{(1)}_{\ell=1, m=0} = 0}$

- extra space - Please state which problem you are working on !

$\ell = 1, m = \pm 1$ degenerate.

consider matrix elements:

$$\begin{pmatrix} \langle \ell=1\ m=+1| \\ \langle \ell=1\ m=-1| \end{pmatrix}^T H_1 \begin{pmatrix} |\ell=1\ m=+1\rangle \\ |\ell=1\ m=-1\rangle \end{pmatrix}$$

need $\langle \ell=1; m=-1| L_-^2 |\ell=1, m=+1\rangle$

$$= \hbar\sqrt{2}\ \langle \ell=1\ m=-1| L_- |\ell=1, m=0\rangle$$

$$= \hbar^2\, 2\ \langle \ell=1, m=-1 | \ell=1, m=-1\rangle$$

$$= 2\hbar^2$$

$$\langle \ell=1, m=+1 | L_+^2 | \ell=1, m=-1\rangle$$

$$= 2\hbar^2 \quad \text{(by hermitian conjugation)}$$

all other matrix elements vanish due to

$$L_+|\ell=1, m=1\rangle = 0 \ \ \& \ \ L_-|\ell=1, m=-1\rangle = 0$$

perturbation: $\begin{pmatrix} 0 & -\frac{\Delta\hbar^2}{I} \\ -\frac{\Delta\hbar^2}{I} & 0 \end{pmatrix}$ $\quad E^{(1)}$ eigenvalues:

$$\boxed{E^{(1)}_{\ell=1, m=\pm1} = \pm\frac{\Delta\hbar^2}{I}}$$

QUESTION 3: [40 points]
Consider two particles with mass m coordinates x_1 and x_2 moving in a one one dimensional box with potential

$$V(x) = \begin{cases} \infty & x < 0 \\ 0 & 0 \leq x \leq L \\ \infty & x > L \end{cases} \tag{0.11}$$

a) Assuming the particles are distinguishable what are the energies and degeneracies of the three lowest energy levels ?

b) Assume that the two particles are identical spin 1/2 fermions. If they are in the singlet state, what is the lowest energy state and what is its energy and ~~generally~~ ? degeneracy

c) If they are in the triplet state what is the lowest energy state and its energy and degeneracy?

d) We add an interaction for the particles, which does not act on the spin but has the form

$$H_1(x_1, x_2) = g\delta(x_1 - x_2) \tag{0.12}$$

Calculate the change in the lowest energy state for the triplet and singlet case to first order in perturbation theory.

e) For the lowest energy triplet states calculate the second order perturbation theory contribution to the lowest energy level.

a) single particle $H = \frac{p^2}{2m}$ eigenstates: $\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L} x\right)$

↑ Normalization.

Energies EV $E = \frac{\hbar^2}{2m} \frac{n^2\pi^2}{L^2}$

2 particles: $|n_1, n_2\rangle$ $E_{n_1 n_2} = \underbrace{\frac{\hbar^2 \pi^2}{2m L^2}}_{\bar{E}} (n_1^2 + n_2^2)$

distinguishable: $n_1 = 1$ $n_2 = 1$ $E = \bar{E} \times 2$ deg 1

$n_2 = 2, n_1 = 1$ & $n_1 = 1$ $n_2 = 2$ $E = \bar{E} \times 5$ deg 2

$E = \bar{E} \times 8$ deg 1

$n_2 = 2 \cdot n_1 = 2$

- extra space - Please state which problem you are working on !

b) singlet spin state

$|singlet\rangle = \frac{1}{\sqrt{2}}(|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle)$

antisymmetric $\Rightarrow$ spatial wavefunction is symmetric

$\psi_{n_1=1}(x_1)\,\psi_{n_2=1}(x_2)$ is symmetric

$\Rightarrow E = 2\bar{E}$ deg. 1

c) triplet symmetric spin state $|l=1, m=0\rangle$
$|l=1\ m=-1\rangle$
$|l=1\ m=+1\rangle$
degeneracy 3

spatial wavefunction:

$$\psi_{Spat}\ \frac{1}{\sqrt{2}}\left(\psi_{n_1=1}(x_1)\psi_{n_2=2}(x_2) - \psi_{n_1=2}(x_1)\,\psi_{n_2=1}(x_2)\right)$$

$E = \cancel{5}\ 3\ \bar{E}$ degeneracy $\cancel{5}$ 3

- extra space - Please state which problem you are working on !

3d) Interaction is diagonal in spin

$H_1 = g\,\delta(x_1 - x_2)$

can use no degenerate perturbation theory for both triplet & singlet.

General formula:

$$|\psi_n^{(1)}\rangle = -\sum_{m\neq n} \frac{|\psi_m^{(0)}\rangle}{E_m^{(0)} - E_n^{(0)}} \langle\psi_m^{(0)}|H_1|\psi_n^{(0)}\rangle$$

For triplet: matrix element

$$\langle\psi_{n=(1,2)}|H_1|\psi_m^{(0)}\rangle$$

$$= \int dx_1\, dx_2\, \frac{1}{\sqrt{2}}\left(\psi_{n=1}(x_1)\psi_{n=2}(x_2) - \psi_{n=2}(x_1)\psi_{n=1}(x_2)\right) g\,\delta(x_1 - x_2)\, \psi_m(x_1, x_1)$$

$= 0$ due to antisymmetry if $x_1 = x_2$ by δ-function

$$\Rightarrow |\psi^{(1)}_{\substack{\text{triplet}\\ n=(1,2)}}\rangle = 0$$

- extra space - Please state which problem you are working on !

For <u>singlet</u>:

$$\psi_{n_1 n_2}(x_1 x_2) = \frac{1}{\sqrt{2}}\Big(\psi_{n=n_1}(x_1) \cdot \psi_{n=n_2}(x_2) + \psi_{n=n_2}(x_1)\ \psi_{n=n_1}(x_1)\Big)$$

$$E = \overline{E}(n_1^2 + n_2^2)$$

$$\psi_{11}(x_1 x_2) = -\sum_{n_1 n_2 \neq (1,1)} \frac{\psi_{n_1 n_2}(x_1 x_2)}{\overline{E}(n_1^2+n_2^2-2)} \langle \psi_{n_1 n_2} | H_1 | \psi_{11} \rangle$$

$$\langle \psi_{n_1 n_2} | H_1 | \psi_{11} \rangle = \left(\frac{1}{\sqrt{2}}\right) \int dx_1 dx_2 \Big(\psi_{n_1}(x_1)\psi_{n_2}(x_2) + \psi_{n_1}(x_2)\psi_{n_2}(x_1)\Big)\ g\,\delta(x_1 + x_2)\ \psi_1(x_1)\psi_1(x_2)$$

$$= \sqrt{2} g \int_0^L dx\ \psi_{n_1}(x)\psi_{n_2}(x)(\psi_1(x))^2$$

$$= \sqrt{2} g \int_0^L dx \left(\frac{2}{L}\right)^2 \sin\left(\frac{n_1 \pi}{L}x\right) \sin\left(\frac{n_2 \pi}{L}x\right) \sin^2 \frac{\pi}{L}x$$

good enough for full credit

- extra space - Please state which problem you are working on !

e) triplet state antisymmetric wavefunct.
2nd order contribution from
$\langle n | H^{(1)} | m \rangle$ has the form

$$\int dx_1 dx_2 \frac{1}{\sqrt{2}}\left(\psi_{n=1}(x_1)\psi_{n=2}(x_2) - \psi_{n=2}(x_1)\psi_{n=1}(x_2)\right)$$

$$\cdot\, g\,\delta(x_1 - x_2)\,\psi_{any}(x_1, x_2)$$

δ function sets $x_1 = x_2$ & matrix
element vanishes.

PHYSICS 221B

Practice Final

Real exam: Thursday March 20th, 2014, at 11.30am - 2.30pm. Room: TBA

- Please write clearly
- If you separate pages please print your name on every page used, including this one;
- Make clear which question and which part you are answering on each page.
- No core-dumps please !
- No books, notes, computers, or calculators are allowed during the exam;
- Please turn off all electronic devices.

Good Luck !!

question	possible points	achieved points
1.	40	
2.	30	
3.	30	
4.	40	
5.	40	
Total	180	

Some possibly useful formulas

1. Harmonic oscillator for a Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 = \hbar\omega(a^\dagger a + \frac{1}{2}) \tag{0.1}$$

with

$$a = \frac{1}{\sqrt{2m\hbar\omega}}(m\omega x + ip) \qquad\qquad [a, a^\dagger] = 1 \tag{0.2}$$

2. The angular momentum algebra is $[J^1, J^2] = i\hbar J^3$, and two cyclic permutations thereof. The corresponding ladder operators are defined to be $J^\pm = J^1 \pm iJ^2$, and act by

$$J^\pm|j,m\rangle = \hbar\sqrt{j(j+1) - m(m\pm 1)}\,|j, m\pm 1\rangle \tag{0.3}$$

3. spherical Bessel functions j_l

$$j_0(x) = \frac{\sin x}{x} \tag{0.4}$$

$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x} \tag{0.5}$$

4. Spherial harmonics

$$Y_0^0 = \frac{1}{\sqrt{4\pi}} \tag{0.6}$$

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta \tag{0.7}$$

$$Y_1^{\pm 1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\phi} \tag{0.8}$$

5. Scattering formula

$$\begin{aligned} e^{i\mathbf{k}\cdot\mathbf{r}} &= \sum_{\ell=0}^{\infty} i^\ell(2\ell+1)j_\ell(kr)P_\ell(\cos\theta) \\ \psi_{\mathbf{k}}(\mathbf{r}) &= \frac{1}{(2\pi)^{3/2}}\left[e^{i\mathbf{k}\cdot\mathbf{r}} + f(\mathbf{k}', \mathbf{k})\frac{e^{ikr}}{r}\right] \\ f(\mathbf{k}', \mathbf{k}) &= \sum_{\ell=0}^{\infty}(2\ell+1)\frac{e^{2i\delta_\ell} - 1}{2ik}P_\ell(\cos\theta) \end{aligned} \tag{0.9}$$

6. phase shift for spherical symmetric potential, 1st Born approximation

$$\delta_l = -k \int_0^\infty dr r^2 U(r) (j_l(kr))^2 \tag{0.10}$$

7. Possible useful integrals:

$$\int_0^{2\pi} d\phi \, e^{ia \sin\phi} = 2\pi J_0(|a|) \tag{0.11}$$

$$\int_0^{\pi} d\theta \sin\theta J_0(b \sin\theta) = \frac{2 \sin b}{b} \tag{0.12}$$

where J_0 is the Bessel function of the first kind.

QUESTION 1: [40 points]

Consider a particle of mass m moving in one dimension, subject to a attractive delta function potential $V_0(x) = -\alpha\delta(x)$.

a) Verify that there is a bound state

$$\psi_0^{(0)} = \sqrt{\kappa}e^{-\kappa|x|}, \qquad \kappa = \frac{m\alpha}{\hbar^2} \tag{0.13}$$

with energy

$$E_0^{(0)} = -\frac{\hbar^2\kappa^2}{2m} \tag{0.14}$$

(You can assume without proof that this is the only state with negative energy).

b) There are also continuous (scattering states) with positive energy. Verify that the state

$$\psi_{-,k}(x) = \frac{1}{\sqrt{\pi}}\sin kx \tag{0.15}$$

is an eigenstate of the Hamiltonian $H_0 = \frac{p^2}{2m} + V_0$. (Note there are also states which are even under parity $x \to -x$, which are modified by the presence of V_0).

c) We now introduce a small uniform electric field which leads an additional contribution to the potential

$$V_1 = -eEx \tag{0.16}$$

Treating V_1 as a perturbation. Calculate the correction of the bound state energy E_0 to second order in perturbation theory.

Hint: In the formulae for perturbation theory the discrete sum gets replaced by an integral for a continuum. Find an argument why only the states found in b) contribute (i.e. the states with even parity do not contribute).

d) Calculate the polarizability for the ground state.

QUESTION 2: [30 points]

Consider the scattering of a spinless particle of mass m from a diatomic molecule. The incoming momentum is $\vec{p} = \hbar k \hat{e}_z$ Assume that the molecule is much heavier than the scattering particle and that there is no recoil. The two atoms in the molecule are aligned along the y-axis and localized at $y = b$ and $y = -b$. The potential the particle feels in the presence of the molecule can be modeled by delta functions:

$$V(\vec{x}) = \alpha\Big(\delta(y-b)\delta(x)\delta(z) + \delta(y+b)\delta(x)\delta(z)\Big)$$

a) Calculate the scattering amplitude in the first Born approximation.

b) Calculate the differential cross section from a) (Express the result in terms of the scattering angles).

c) Calculate the total cross section. You can **either** do the integrals exactly **or** calculate the total cross section to order k^2 (inclusive) in the small k limit.

QUESTION 3: [30 points]

A two state system is described by the following Hamiltonian

$$H = H_0 + V(t)$$

With a time independent H_0 and a two orthonormal basis vectors satisfying

$$H_0 \mid 1\rangle = \epsilon_1 \mid 1\rangle, \quad H_0 \mid 2\rangle = \epsilon_2 \mid 2\rangle$$

The perturbation satisfies

$$V(t) \mid 1\rangle = \hbar\omega_1 e^{-i\omega t} \mid 2\rangle, \quad V(t) \mid 2\rangle = \hbar\omega_1 e^{i\omega t} \mid 1\rangle$$

a) Find the eigenvalues and eigenvectors of H

b) Solve the time dependent Schrödinger equation for $t > 0$ for a state with initial condition

$$\mid \psi(t = 0)\rangle = \mid 1\rangle$$

c) Calculate the probability to find the system at time $t > 0$ in the state $\mid 2\rangle$.

QUESTION 4: [40 points]

A particle is scattered by a spherical symmetric potential at energies which are low enough so that only the phase shifts δ_0 and δ_1 are nonzero. (For part a)-c) treat δ_0, δ_1 as given).

a) Show that the differential cross section is of the form

$$\frac{d\sigma}{d\Omega} = A + B\cos\theta + C\cos^2\theta$$

b) Determine A, B, C in terms of the phase shifts

c) Calculate the total cross section in terms of A, B, C.

d) Consider a very weak and short range potential (which behaves not worse than $1/r$ at the origin). Estimate the k dependence of δ_0 and δ_1 in the limit $k \to 0$.

QUESTION 5: [40 points]

Consider the one dimensional harmonic oscillator with Hamiltonian

$$H_0 = \frac{p^2}{2m} + \frac{1}{2} m\, \omega^2 x^2$$

At time $t > 0$ the following perturbation is turned on

$$H'(t) = \alpha\, x\, e^{-\frac{t}{\tau}}$$

a) If at time $t < 0$ the system is in its ground state (of H_0) calculate to first order in time dependent perturbation theory the probability that the system is found at time $t > 0$ in the first excited state.

b) If at time $t < 0$ the system is in the first excited state (of H_0) calculate to first order in time dependent perturbation theory the probability that the system is found at time $t > 0$ in the ground state.

c) If the system at time $t < 0$ is in the found state of H_0, at which order in perturbation theory would you expect to find a nonzero transition probably to the second excited state (Why?). Calculate this probability.

d) For the harmonic oscillator with Hamiltonian H_0 above, give an example of an adiabatic change and a sudden change. What is the time scale which is used to decide whether an adiabatic or sudden change approximation is appropriate ?

If the system is in the ground state at time $t = 0$ describe (without calculation) how the state evolves at later times for the two cases.

Quantum Mechanics: 221B, Practice Final

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1 1D δ Potential

(a) Bound State

If $V(x) = -\alpha\delta(x)$

$$\begin{aligned} -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) - \alpha\delta(x)\psi(x) &= E\psi(x) \\ \frac{d^2}{dx^2}\psi(x) - \beta^2\psi(x) &= -\frac{2m\alpha}{\hbar^2}\delta(x)\psi(x) \end{aligned} \tag{1}$$

Where $\beta^2 = -\frac{2mE}{\hbar^2}$. Apart from the origin, this is a first order homogeneous equation. The general solution is:

$$\psi(x) = \begin{cases} A\exp(-\beta x) & x > 0 \\ B\exp(\beta x) & x < 0 \end{cases} \tag{2}$$

Here we used the condition that the wave function should be normalizable, so $\psi(x) \to 0$, as $x \to \pm\infty$. A, B are constants which will be determined later.

Now use the connecting condition:

$$\begin{aligned} \psi(0-) &= \psi(0+) \\ \frac{d}{dx}\psi(0+) - \frac{d}{dx}\psi(0-) &= -\frac{2m\alpha}{\hbar^2}\psi(0) \end{aligned} \tag{3}$$

Take the general solution into the above equation, we first find $A = B$, and

$$-2A\beta = -\frac{2m\alpha}{\hbar^2}A \tag{4}$$

So, $\beta = \frac{m\alpha}{\hbar^2}$, and the bound state energy is $E = -\frac{m\alpha^2}{2\hbar^2}$.
Using normalization condition, the constant is found to be $A = \sqrt{\beta}$.

So, to sum up, there is only one bound state with energy $E = -\frac{m\alpha^2}{2\hbar^2}$, and $\psi(x) = \sqrt{\beta}\exp(-\beta|x|)$

(b) Continuum

It's easy to see that $p^2\psi_{-k}(x) \propto \sin(kx) \propto \psi_{-k}(x)$, and that $\psi_{-k}(0) = 0$, so the V_0 term vanishes for this state.

(c) Electric Field Perturbation

First, notice that, for a discrete spectrum

$$\Delta E_n = \lambda V_{nn} + \lambda^2 \sum_{k \neq n} \frac{|\langle n| V |k\rangle|^2}{E_n^{(0)} - E_k^{(0)}}$$

we are considering a case where $|n\rangle$ will be even, and $V \propto x$. $V|k\rangle$ must therefore be even, so $|k\rangle$ must be odd. This tells us that the first order correction vanishes. Generalizing to the continuum,

$$\Delta E_n = \int \frac{dk}{\sqrt{2\pi}} \frac{|\langle n| V |k\rangle|^2}{E_0^{(0)} - \frac{\hbar^2 k^2}{2m}} = -\frac{2me^2E^2}{\hbar^2\kappa^2\sqrt{2\pi}} \int dk \, \frac{|\langle n| \hat{x} |k\rangle|^2}{1 + (k/\kappa)^2}$$

We have that

$$\begin{aligned}
\langle n|\,\hat{x}\,|k\rangle &= \int dx\,\sqrt{\kappa}e^{-\kappa|x|}x\frac{1}{\sqrt{\pi}}\sin kx = \sqrt{\frac{\kappa}{\pi}}\int dx\,e^{-\kappa|x|}x\sin kx\\
&= 2\sqrt{\frac{\kappa}{\pi}}\int_0^\infty dx\,e^{-\kappa x}x\sin kx = 2\sqrt{\frac{\kappa}{\pi}}\int_0^\infty \frac{1}{k}du\,e^{-(\kappa/k)u}u/k\sin u\\
&= 2\sqrt{\frac{\kappa}{\pi}}\frac{1}{k^2}\int_0^\infty du\,e^{-(\kappa/k)u}u\sin u = 2\sqrt{\frac{\kappa}{\pi}}\frac{1}{k^2}\partial_\alpha\int_0^\infty du\,e^{-\alpha u}\sin u\Big|_{\alpha=\kappa/k}\\
&= 2\sqrt{\frac{\kappa}{\pi}}\frac{1}{k^2}\partial_\alpha\,\frac{1}{\alpha^2+1}\Big|_{\alpha=\kappa/k} = 2\sqrt{\frac{\kappa}{\pi}}\frac{1}{k^2}\,\frac{-2\alpha}{(\alpha^2+1)^2}\Big|_{\alpha=\kappa/k} = -2\sqrt{\frac{\kappa}{\pi}}\frac{2k\kappa}{(\kappa^2+k^2)^2}
\end{aligned}$$

so

$$\begin{aligned}
\Delta E_n &= -\frac{8\kappa m e^2E^2}{\pi\hbar^2\kappa^2\sqrt{2\pi}}\int dk\,\frac{1}{1+(k/\kappa)^2}\frac{4k^2\kappa^2}{(\kappa^2+k^2)^4}\\
&= -\frac{8\kappa m e^2E^2}{\pi\hbar^2\kappa^6\sqrt{2\pi}}\int dk\,\frac{1}{1+(k/\kappa)^2}\frac{4(k/\kappa)^2}{(1+(k/\kappa)^2)^4}\\
&= -\frac{32 m e^2E^2}{\pi\hbar^2\kappa^5\sqrt{2\pi}}\int dk\,\frac{(k/\kappa)^2}{(1+(k/\kappa)^2)^5}
\end{aligned}$$

(d) Polarizability

$$\alpha = -\frac{2}{E^2}\Delta E_n^{(2)}$$

2 Diatomic Scattering of Spinless Particle

(a) First Born

$$U = \frac{2m}{\hbar^2}V = \tilde{\alpha}\delta(x)\delta(z)\left[\delta(y-b)+\delta(y+b)\right]$$

$$\begin{aligned}
f(\boldsymbol{k},\boldsymbol{k}') &= -\frac{(2\pi)^{3/2}}{4\pi}\int d^3\boldsymbol{y}e^{-i\boldsymbol{k}'\cdot\boldsymbol{y}}U(\boldsymbol{y})e^{ik\cdot\boldsymbol{y}} = -\frac{2m\alpha}{\hbar^2}\frac{(2\pi)^{3/2}}{4\pi}\int dy e^{i(k_y-k_y')y}\left[\delta(y-b)+\delta(y+b)\right]\\
&= -\frac{2m\alpha}{\hbar^2}\frac{(2\pi)^{3/2}}{4\pi}\left[e^{i(k_y-k_y')b}+e^{-i(k_y-k_y')b}\right] = -\frac{4m\alpha}{\hbar^2}\frac{(2\pi)^{3/2}}{4\pi}\cos(k_y-k_y')b
\end{aligned}$$

(b) Cross Section

Using $\boldsymbol{p} = \hbar k\hat{\boldsymbol{z}}$, $k_y = 0$, so $k_y' = k\sin\theta\cos\phi$, and

$$\frac{d\sigma}{d\Omega} = |f|^2 = \frac{8\pi^2m^2\alpha^2}{\hbar^4}\cos^2(kb\sin\theta\cos\phi)$$

(c) Total Cross Section

$$\sigma_t = \frac{8\pi m^2\alpha^2}{\hbar^4}\int\cos^2(kb\sin\theta\cos\phi)\,\sin\theta\,d\theta d\phi \to \frac{32\pi^3m^2\alpha^2}{\hbar^4}$$

as $b\to 0$.

3 Two State System

(a) Eigenvalues and Eigenvectors

Put $U = \begin{bmatrix} e^{i\omega t/2} & 0 \\ 0 & e^{-i\omega t/2} \end{bmatrix}$, so that $U^\dagger H U = \begin{bmatrix} \epsilon_1 & \epsilon' \\ \epsilon' & \epsilon_2 \end{bmatrix}$, with $\epsilon' = \hbar\omega_1$. This has eigenvalues λ given by

$$0 = (\epsilon_1 - \lambda)(\epsilon_2 - \lambda) - \epsilon'^2 = \lambda^2 - (\epsilon_1 + \epsilon_2)\lambda + (\epsilon_1\epsilon_2 - \epsilon'^2)$$

or

$$\lambda_\pm = \frac{\epsilon_1 + \epsilon_2 \pm \sqrt{(\epsilon_1 + \epsilon_2)^2 - 4(\epsilon_1\epsilon_2 - \epsilon'^2)}}{2} = \frac{\epsilon_1 + \epsilon_2 \pm \sqrt{(\epsilon_1 - \epsilon_2)^2 + 4\epsilon'^2}}{2}$$

For the eigenvector,

$$\lambda_\pm \begin{bmatrix} a \\ b \end{bmatrix} = U^\dagger H U \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a\epsilon_1 + \epsilon' b \\ \cdots \end{bmatrix}$$

or

$$\lambda_\pm - \epsilon_1 = \frac{b}{a}\epsilon' \quad \text{or} \quad g_\pm = \frac{b}{a} = \frac{\lambda_\pm - \epsilon_1}{\epsilon'}$$

I.e., the eigenvectors are $U|\phi_\pm\rangle$, with

$$|\phi_\pm\rangle = \frac{1}{\sqrt{2 + 2g_\pm^2}} \begin{bmatrix} 1 \\ g_\pm \end{bmatrix}$$

(b) Solution to Time-dependent Schrödinger Equation

$$i\hbar\partial_t |\psi(t)\rangle = H(t)|\psi(t)\rangle$$

or

$$i\hbar U^\dagger \partial_t |\psi(t)\rangle = U^\dagger H(t) U U^\dagger |\psi(t)\rangle$$

but

$$\partial_t U^\dagger(t)|\psi(t)\rangle = \left(\partial_t U^\dagger(t)\right)|\psi(t)\rangle + U^\dagger(t)\partial_t|\psi(t)\rangle$$

so that

$$U^\dagger(t)\partial_t|\psi(t)\rangle = \partial_t U^\dagger(t)|\psi(t)\rangle - \frac{i\omega}{2}\sigma_z U^\dagger(t)|\psi(t)\rangle$$

especially

$$i\hbar\partial_t U^\dagger|\psi(t)\rangle = \left[-\frac{\hbar\omega}{2}\sigma_z + U^\dagger H(t) U\right] U^\dagger|\psi(t)\rangle$$

and therefore

$$U^\dagger|\psi(t)\rangle = \left[e^{-i(\lambda_+/\hbar - \omega/2)t}|\phi_+\rangle\langle\phi_+| + e^{-i(\lambda_-/\hbar + \omega/2)t}|\phi_-\rangle\langle\phi_-|\right]|\psi(0)\rangle$$

If $|\psi(0)\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, then

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}U(t)\left[\frac{1}{\sqrt{1+g_+^2}}e^{-i(\lambda_+/\hbar - \omega/2)t}|\phi_+\rangle + \frac{g_-}{\sqrt{1+g_-^2}}e^{-i(\lambda_-/\hbar + \omega/2)t}|\phi_-\rangle\right]$$

(c) Finally 2

This is just substitution:

$$\begin{aligned}\langle 2|\psi(t)\rangle &= \frac{e^{-i\omega t/2}}{\sqrt{2}}\left[\frac{1}{\sqrt{1+g_+^2}}e^{-i(\lambda_+/\hbar - \omega/2)t}\langle 2|\phi_+\rangle + \frac{g_-}{\sqrt{1+g_-^2}}e^{-i(\lambda_-/\hbar + \omega/2)t}\langle 2|\phi_-\rangle\right] \\ &= \frac{e^{-i\omega t/2}}{\sqrt{2}}\left[\frac{g_+}{1+g_+^2}e^{-i(\lambda_+/\hbar - \omega/2)t} + \frac{g_-}{1+g_-^2}e^{-i(\lambda_-/\hbar + \omega/2)t}\right]\end{aligned}$$

4 Spherical Symmetric Potential

(a) Different Cross Section: f

$$f(\theta,\phi) = \frac{1}{2ik}\sum_l (2l+1)(e^{2i\delta_l}-1)P_l(\cos\theta) \doteq \frac{1}{2ik}\left[(e^{2i\delta_0}-1)+3(e^{2i\delta_1}-1)\cos\theta\right]$$

It clear that squaring this will gives terms like the claim; see the next part for their determination.

(b) Different Cross Section: Revisited

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= |f|^2 \doteq \frac{1}{4k^2}\left[(e^{2i\delta_0}-1)+3(e^{2i\delta_1}-1)\cos\theta\right]\left[(e^{-2i\delta_0}-1)+3(e^{-2i\delta_1}-1)\cos\theta\right] \\ &= \frac{1}{2k^2}\left[(1-\cos(2\delta_0)+9(1-\cos(2\delta_1)\cos^2\theta+3\left((e^{2i\delta_0}-1)(e^{-2i\delta_1}-1)+(e^{-2i\delta_0}-1)(e^{2i\delta_1}-1)\right)\cos\theta\right]\end{aligned}$$

Next:

$$\begin{aligned}&(e^{2i\delta_0}-1)(e^{-2i\delta_1}-1)+(e^{-2i\delta_0}-1)(e^{2i\delta_1}-1) \\ &\quad = e^{2i\delta_0}(e^{-2i\delta_1}-1)-(e^{-2i\delta_1}-1)+e^{-2i\delta_0}(e^{2i\delta_1}-1)-(e^{2i\delta_1}-1) \\ &\quad = e^{2i(\delta_0-\delta_1)}-e^{2i\delta_0}-e^{-2i\delta_1}+1+e^{-2i(\delta_0-\delta_1)}-e^{-2i\delta_0}-e^{2i\delta_1}+1 \\ &\quad = 2\left[1+\cos 2(\delta_0-\delta_1)-\cos(2\delta_0)-\cos(2\delta_1)\right]\end{aligned}$$

so that

$$A = \frac{1-\cos(2\delta_0)}{2k^2}$$

$$B = \frac{3}{k^2}\left[1+\cos 2(\delta_0-\delta_1)-\cos(2\delta_0)-\cos(2\delta_1)\right]$$

and

$$C = \frac{9}{2k^2}(1-\cos(2\delta_1))$$

(c) Cross Section

$$\begin{aligned}\sigma_t &= \int\frac{d\sigma}{d\Omega}\,d\Omega = A\int d\Omega + B\int\cos\theta\,d\Omega + C\int\cos^2\theta\,d\Omega \\ &= 4\pi A + 2\pi B\int_1^{-1}\cos\theta\,d[\cos\theta] + 2\pi C\int_1^{-1}\cos^2\theta\,d[\cos\theta] = 4\pi A + \frac{4}{3}\pi C\end{aligned}$$

(d) Low Momentum Limit

We use the 1st Born approximation

$$\delta_l = -k\int_0^\infty r^2\,dr\,U(r)(j_l(kr))^2$$

For δ_0, $j_0 = \frac{\sin(kr)}{kr}$, so

$$\delta_0 = -k\int_0^\infty r^2\,dr\,U(r)\frac{\sin^2(kr)}{k^2r^2} \to -k\int_0^\infty r^2\,dr\,U(r) = -\frac{k}{4\pi}\int dV\,U$$

(it goes like $-\frac{k}{4\pi}$ times the total interaction strength). For δ_1, $j_1 = \frac{\sin kr}{k^2r^2} - \frac{\cos kr}{kr} \to \frac{1}{kr} - \frac{1-k^2r^2}{kr} = -kr$ so that

$$\delta_1 \to -k\int_0^\infty r^2\,dr\,U(r)k^2r^2 = -k^3\int_0^\infty r^4\,dr\,U(r)$$

5 Perturbed SHO

(a) First Excited State

$$c_1^{(1)}(t) = -\frac{i}{\hbar}\int_0^t e^{i\omega_{10}t'} V_{10}(t')\,dt' = -\frac{i\alpha \langle 1|\,x\,|0\rangle}{\hbar}\int_0^t e^{(i\omega_{10}-\tau^{-1})t'}\,dt' = -\frac{\alpha \langle 1|\,x\,|0\rangle}{\hbar\omega - \hbar\tau^{-1}}\left[1 - e^{(i\omega-\tau^{-1})t}\right]$$

and

$$a + a^\dagger = 2m\omega\sqrt{\frac{1}{2m\hbar\omega}}x \quad \text{or} \quad x = \sqrt{\frac{2m\omega}{\hbar}}(a + a^\dagger)$$

$$\begin{aligned}\langle 1|\,x\,|0\rangle = \langle 0|\,ax\,|0\rangle &= \sqrt{\frac{2m\omega}{\hbar}}\,\langle 0|\,a\left[a^\dagger + a\right]|0\rangle \\ &= \sqrt{\frac{2m\omega}{\hbar}}\,\langle 0|\,aa^\dagger\,|0\rangle = \sqrt{\frac{2m\omega}{\hbar}}\end{aligned}$$

We can simplify a little:

$$\begin{aligned}\left|1 - e^{(i\omega-\tau^{-1})t}\right|^2 &= (1-\cos(\omega t)e^{-t/\tau})^2 + \sin^2(\omega t)e^{-2t/\tau} \\ &= 1 - 2\cos(\omega t)e^{-t/\tau} + \cos^2(\omega t)e^{-2t/\tau} + \sin^2(\omega t)e^{-2t/\tau} = 1 + e^{-2t/\tau} - 2\cos(\omega t)e^{-t/\tau}\end{aligned}$$

To first order in perturbation theory,

$$\left|c_1^{(1)}(t)\right|^2 = -\frac{2m\omega}{\hbar}\frac{\alpha^2}{\left[\hbar\omega - \hbar\tau^{-1}\right]^2}\left|1 - e^{(i\omega-\tau^{-1})t}\right|^2 = -\frac{2m\omega}{\hbar}\frac{\alpha^2}{\left[\hbar\omega - \hbar\tau^{-1}\right]^2}\left(1 + e^{-2t/\tau} - 2\cos(\omega t)e^{-t/\tau}\right)$$

(b) Ground State

$$c_n^{(1)}(t) = -\frac{i}{\hbar}\int_0^t e^{i\omega_{n0}t'} V_{n0}(t')\,dt' = -\frac{i\alpha \langle n|\,x\,|0\rangle}{\hbar}\int_0^t e^{(i\omega_{n0}-\tau^{-1})t'}\,dt'$$

For $n \neq 1$

$$\langle n|\,x\,|0\rangle \propto \langle 0|\,a^n x\,|0\rangle \propto \langle 0|\,a^n\left[a^\dagger + a\right]|0\rangle = 0$$

To first order in perturbation theory,

$$P_0(t) = 1 - \sum_n |c_n^{(1)}(t)|^2 = 1 - |c_1^{(1)}(t)|^2$$

(c) Second Excited State

We use second order perturbation theory

$$
\begin{aligned}
c_2^{(2)}(t) &= \left(-\frac{i}{\hbar}\right)^2 \sum_n \int_0^t dt' \int_0^{t'} dt''\, e^{i\omega_{2m}t'} V_{2m}(t') e^{i\omega_{m0}t''} V_{m0}(t') \\
&= \left(-\frac{i}{\hbar}\right)^2 \int_0^t dt' \int_0^{t'} dt''\, e^{i\omega_{21}t'} V_{21}(t') e^{i\omega_{10}t''} V_{10}(t') \\
&= -\frac{\alpha^2 \langle 2|\, x\,|1\rangle \langle 1|\, x\,|0\rangle}{\hbar^2} \int_0^t dt' \int_0^t dt'' e^{(i\omega_{21}-\tau^{-1})t'} e^{(i\omega_{10}-\tau^{-1})t''} \\
&= -\frac{\alpha^2 \langle 2|\, x\,|1\rangle \langle 1|\, x\,|0\rangle}{\hbar^2} \frac{1}{i\omega-\tau^{-1}} \int_0^t dt' e^{(i\omega-\tau^{-1})t'} \left[1-e^{(i\omega-\tau^{-1})t'}\right] \\
&= -\frac{\alpha^2 \langle 2|\, x\,|1\rangle \langle 1|\, x\,|0\rangle}{\hbar^2} \frac{1}{i\omega-\tau^{-1}} \int_0^t dt' \left[e^{(i\omega-\tau^{-1})t'} - e^{2(i\omega-\tau^{-1})t'}\right] \\
&= -\frac{\alpha^2 \langle 2|\, x\,|1\rangle \langle 1|\, x\,|0\rangle}{\hbar^2} \frac{1}{(i\omega-\tau^{-1})^2} \left(\left[1-e^{(i\omega-\tau^{-1})t}\right] - \frac{1}{2}\left[1-e^{2(i\omega-\tau^{-1})t}\right]\right) \\
&= -\frac{\alpha^2 \langle 2|\, x\,|1\rangle \langle 1|\, x\,|0\rangle}{\hbar^2} \frac{1}{2\,(i\omega-\tau^{-1})^2} \left[1-e^{(i\omega-\tau^{-1})t}\right]
\end{aligned}
$$

and we already have that

$$
x = \sqrt{\frac{2m\omega}{\hbar}}(a+a^\dagger)
$$

and $\langle 1|\, x\,|0\rangle = \sqrt{\frac{2m\omega}{\hbar}}$. We can also get

$$
\begin{aligned}
\langle 2|\, x\,|1\rangle &= \sqrt{\frac{2m\omega}{\hbar}} \langle 1| \left[aa^\dagger + a^2\right] |1\rangle \\
&= \sqrt{\frac{2m\omega}{\hbar}} \langle 1| \left[1 + a^\dagger a\right] |1\rangle = 2\sqrt{\frac{2m\omega}{\hbar}}
\end{aligned}
$$

(d) Adiabatic vs. Sudden

In the limit that $\tau \to 0$, the exponential factor becomes a step function. For $t > 0$, we get the full αx potential, and for $t < 0$, we get none. Alternatively, letting $\tau \to 0$, we get a very slow turn on, and consequential adiabatic change.

In the adiabatic limit, the eigenvalues of $H(t)$ change smoothly with time, and the system remains in the same states. E.g., if it started in the ground state, it will follow that state as it changes smoothly all the way to the end of the adiabatic change. In the sudden limit, we imagine that the last state of the initial system acts as the initial state of the next system, and that the changeover region essentially does not affect the system's evolution.

The time scale is set by the energy separation of the levels of the system (i.e., ω in this case). If the time scale is large compared to ω^{-1}, it is adiabatic; conversely, if it is small, the sudden approximation should hold.