

STATISTICAL PHYSICS 215A

Final Exam – Spring 2014

Wednesday 11 June 2014 from 3pm to 6pm in room PAB-2-434

- **Print your name on every pages used, including this one;**
- Make clear which question you are answering on each page;
- No books, notes, computers, or calculators are allowed during the exam;
- Please turn off cell-phones, iPhones, iPods, iPads, Kindles, and other electronic devices.
- Please write clearly; present your arguments and calculations clearly;
- All five questions below are independent from one another.

Grades

Q1.

Q2.

Q3.

Q4.

Total /55

QUESTION 1 [13 points]

- (a) From the first law of thermodynamics, derive the following relation between the pressure P and the internal energy E , both of which are functions of T, V ,

$$\left(\frac{\partial E}{\partial V}\right)_T = T \left(\frac{\partial P}{\partial T}\right)_V - P$$

- (b) The equation of state of a thermodynamics system is given by $P = \alpha \varepsilon(T)$ where α is a constant and $E = V \varepsilon(T)$. Calculate the temperature dependence of E .
(c) Calculate the entropy $S(T, V)$ for the system whose equation of state is given above.

QUESTION 2 [14 points]

Consider a gas of N hard spheres in a box of volume V . A single sphere excludes a volume v around it. There are no interactions between the spheres, except for the constraints of the hard-core exclusion. The center of mass of a single sphere is free to move through the volume available to it and obeys the non-relativistic relation between energy, momentum and mass.

- (a) Calculate the entropy S , as a function of the total energy E ;
[Hint: It may be convenient to use the Γ -function, satisfying $\Gamma(x+1) = x\Gamma(x)$ along with Sterling's formula $\ln \Gamma(x) = x \ln x - x + \mathcal{O}(\ln x)$. Alternatively, you may wish to use the approximate formula $(V - nv)(V - (N - 1 - n)v) \approx (V - N/2)^2$ for $1 \leq n \leq N - 1$.]
(b) Calculate the equation of state for this gas;
(c) Show that the isothermal compressibility $\kappa_T = -V^{-1}(\partial V / \partial P)|_T$ is always positive.

QUESTION 3 [14 points]

We propose to evaluate the electric current density of electrons which is produced by heating up a metal in the presence of an external electric potential. The potential energy for electrons inside the metal vanishes, while just outside the metal it is $W > 0$. The electrons are considered otherwise non-interacting, and filled up to chemical potential μ with $\mu < W$. We consider the problem at sufficiently low temperature so that μ may be identified with the Fermi energy.

- (a) State the condition on its momentum for an electron to be able to escape from the metal to the outside, as a function of W and μ ;
(b) Derive a general integral representation for the current density I of electrons leaving the metal;
(c) Obtain an approximation of your result in (b) valid for sufficiently low temperatures.

QUESTION 4 [14 points]

(a) For any system of fermions at chemical potential μ and temperature T , show that the probability for finding an occupied state of energy $\varepsilon + \mu$ is the same as that for finding an unoccupied state of energy $\mu - \varepsilon$.

Consider now a system of non-interacting Dirac fermions of spin 1/2 and mass m . One-particle states at momentum \mathbf{k} come in pairs of positive and negative energy,

$$\varepsilon_{\pm}(\mathbf{k}) = \pm \sqrt{m^2 c^4 + \mathbf{k}^2 c^2}$$

for each value of the spin quantum number. At $T = 0$, all negative energy Dirac states are filled (the so-called Dirac-sea), and all positive energy states are empty, so that $\mu(T = 0) = 0$.

- (b) Using the result of (a) compute the chemical potential at arbitrary temperature T .
- (c) Compute (an integral representation for) the mean excitation energy $E(T) - E(0)$
- (d) Evaluate the integral in part (c) for $m = 0$ and evaluate the specific heat C_V ;
- (e) Describe qualitatively the dependence on m in the specific heat at low temperature when $m \neq 0$.

STATISTICAL PHYSICS 215A

Midterm Exam – Spring 2014

Tuesday 6 May 2014 from 1pm to 2;30pm in room PAB-2-434

- Please write clearly;
 - Present your arguments and calculations clearly;
 - All four questions below are independent from one another.
-
- Print your name on every page used, including this one;
 - Make clear which question you are answering on each page;
 - No books, notes, computers, or calculators are allowed during the exam;
 - Please turn off cell-phones, iPhones, iPods, iPads, Kindles, and other electronic devices.

Grades

Q1.

Q2.

Q3.

Total /20

QUESTION 1 [7 points]

When n atoms in a perfect crystal formed by N atoms (where $1 \ll n \ll N$) are displaced from lattice sites inside the crystal to lattice sites on the surface, the crystal becomes imperfect, and has defects. Let $w > 0$ be the energy necessary to displace an atom from the inside to the surface. Assuming equilibrium at temperature T (with $kT \ll w$), determine an approximate formula for the number of displaced sites n in terms of N, w , and T . Neglect any effect due to the change in volume of the crystal. [Hint: begin by evaluating the entropy of the system.]

QUESTION 2 [7 points]

(a) Calculate the partition function $Z(T, V, N)$ of a gas of N non-interacting ultra-relativistic particles. Each particle is devoid of internal degrees of freedom and its energy ε is related to its momentum by the relativistic $\varepsilon = c|\mathbf{p}|$ for constant speed c .

(b) Compute the internal energy E of the system as a function of T, V, N .

(c) Compute the density of states $g(E)$ as a function of total energy E .

QUESTION 3 [6 points]

A cylinder of radius R and length L rotates about its axis with a constant angular velocity ω . Evaluate the normalized density function ρ of an ideal gas enclosed in the cylinder, within the approximation of classical statistical mechanics, and assuming that equilibrium has set in at temperature T . Ignore the effects of gravitation. [Hint: the Hamiltonian that describes the motion in a rotation frame is $H^* = H - \omega L$ where H is the Hamiltonian in the rest frame, and L is the angular momentum.]

STATISTICAL PHYSICS 215A

Final Exam – Spring 2013

Monday 10 June 2013 from 3pm to 6pm in room PAB-2-434

- Please write clearly;
 - Present your arguments and calculations clearly;
 - All four questions below are independent from one another.
-
- Print your name on every page used, including this one;
 - Make clear which question you are answering on each page;
 - No books, notes, computers, or calculators are allowed during the exam;
 - Please turn off cell-phones, iPhones, iPods, iPads, Kindles, and other electronic devices.

Grades

Q1.

Q2.

Q3.

Q4.

Total /65

Some Useful formulas

- The Γ function obeys,

$$\Gamma(\nu + 1) = \nu\Gamma(\nu) \qquad \Gamma(1/2) = \sqrt{\pi} \qquad (0.1)$$

as well as Sterling's formula in the limit of large N ,

$$\ln \Gamma(N + 1) = N \ln N - N + \mathcal{O}(\ln N) \qquad (0.2)$$

with $\Gamma(N + 1) = N!$ for integer N .

- The following integral may come in useful for question 3,

$$\int_{-\infty}^{\infty} \frac{x^2 dx}{(e^x + e^{-x})^2} = \frac{\pi^2}{24} \qquad (0.3)$$

- The Bose-Einstein functions in question 4 are defined by,

$$g_\nu(z) = \frac{1}{\Gamma(\nu)} \int_0^\infty \frac{x^{\nu-1} dx}{z^{-1}e^x - 1} \qquad (0.4)$$

The integral is absolutely convergent for $|z| < 1$ and $\text{Re}(\nu) > 0$, and admits the series expansion,

$$g_\nu(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^\nu} \qquad (0.5)$$

and satisfies the differential recursion relation,

$$zg'_{\nu+1}(z) = g_\nu(z) \qquad (0.6)$$

as well as the following asymptotic expansion near $z = e^{-\alpha}$ for $0 < \alpha \ll 1$,

$$g_{3/2}(e^{-\alpha}) = \zeta(3/2) - 2\sqrt{\pi}\alpha^{\frac{1}{2}} + \mathcal{O}(\alpha) \qquad (0.7)$$

Finally, we have $g_\nu(1) = \zeta(\nu)$, namely the Riemann ζ -function.

QUESTION 1 [16 points]

For paramagnetic materials, the first law of thermodynamics states that $dE = \delta Q + HdM$, where H is the external magnetic field, and M is the magnetization.

(a) Write down the expressions for the specific heat C_M at constant M , and the specific heat C_H at constant H in terms of the internal energy E , temperature T , as well as H and M .

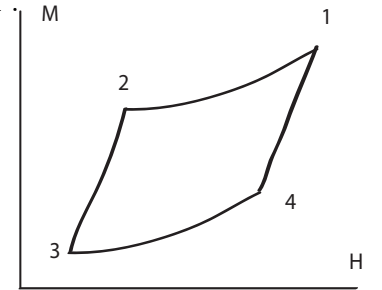
(b) Show that one has,

$$C_H = C_M - \left(\frac{\partial M}{\partial T} \right)_H \left[H - \left(\frac{\partial E}{\partial M} \right)_T \right]$$

(c) Consider now a paramagnetic material obeying Curie's law $M = nDH/T$, for some constant D , and $n = N/V$. This material is magnetized adiabatically from $M = 0$ to a non-zero value of M . Calculate the ratio of the temperatures $T(M)/T(M = 0)$ as a function of M , under the assumption that C_M is constant and that $\partial E/\partial M = 0$ at constant T .

(d) This material is carried around a Carnot cycle with $T_h > T_\ell$.

- 1 \rightarrow 2 M is reduced (demagnetized) isothermally at $T = T_h$;
- 2 \rightarrow 3 M is reduced adiabatically;
- 3 \rightarrow 4 M is increased (magnetized) isothermally at $T = T_\ell$;
- 4 \rightarrow 1 M is increased adiabatically.



Express M_3, M_4 in terms of T_h, T_ℓ, D, C_M, n , and M_1, M_2 .

Show that the efficiency of the cycle is given by $\eta = 1 - T_\ell/T_h$.

QUESTION 2 [16 points]

N indistinguishable quasi-classical particles move in one dimension of space which is a box of length L . The Hamiltonian is that of massless relativistic particles, and is given by,

$$H(\{p_i, q_i\}) = \sum_{i=1}^N c|p_i| \quad (0.8)$$

Use the micro-canonical ensemble, and fix the total energy to be E .

(a) Compute the number of states $\Omega(E, L, N)$ with energy less than E . Compute the number of states $\Omega'(E, L, N, \Delta)$ with energy between E and $E + \Delta$.

[Hint: Evaluate the non-trivial multiple integral recursively in N , using its scaling properties.]

(b) Compute the entropy $S(E, L, N)$. Show that in the thermodynamic limit, the entropy is an extensive quantity.

(c) Derive the relation between energy E and temperature T . Confirm your result using the canonical ensemble.

(d) Find the equation of state $P(N, L, T)$ and compute the specific heat C_L at constant L .

QUESTION 3 [16 points]

(a) Derive a formula for the specific heat C_V of a system of electrons at arbitrary temperature T and chemical potential μ in terms of the FD occupation number

$$f(\varepsilon) = \frac{1}{e^{\beta(\varepsilon-\mu)} + 1} \quad \beta = \frac{1}{kT}$$

and the density of states $D(\varepsilon)$ for one electron per unit volume at energy ε .

(b) Derive a formula for C_V to leading non-zero order for strong degeneracy for general $D(\varepsilon)$.

(c) Derive a formula for C_V to leading non-zero order for weak degeneracy for general $D(\varepsilon)$.

(d) Apply these formulas to a system of free non-relativistic electrons. Do you recover the standard high T result for C_V ? Explain.

QUESTION 4 [17 points]

We consider an ideal non-relativistic Bose-Einstein gas whose constituents have mass m and no internal degrees of freedom. Total particle number N , condensate particle number N_0 , T , E , V , μ and fugacity $z = e^{\beta\mu}$, are related as follows,

$$N - N_0 = \frac{V}{\lambda^3} g_{3/2}(z) \quad E = \frac{3kTV}{2\lambda^3} g_{5/2}(z) \quad (0.9)$$

As usual, we set $\lambda^2 = 2\pi\hbar^2/(mkT)$.

We now consider instead the BE gas confined to a vertical cylinder of height L in the presence of a uniform gravitational acceleration g .

(a) Calculate the critical temperature T_c at which BE condensation sets in under the assumption that the gravitational effects are weak, namely $mgL \ll kT_c$. Express your answer in terms m, g, L and the critical temperature T_c^0 for BE condensation in the absence of gravity.

(b) Show that the effect of gravity produces a discontinuity in the specific heat C_V at the BE transition T_c , and that the value of this discontinuity is given by,

$$\Delta C_V \Big|_{T_c} = -\frac{9}{8\sqrt{\pi}} \zeta\left(\frac{3}{2}\right) Nk \left(\frac{mgL}{kT_c^0}\right)^{\frac{1}{2}} \quad (0.10)$$

STATISTICAL PHYSICS 215A

Final Exam – Spring 2012

Monday 11 June 2012

11:30am – 2:30pm

PAB-2-434

- Please write clearly;
 - Present your arguments and calculations clearly;
 - All five questions below are independent from one another.
-
- Print your name on every page used, including this one;
 - Make clear which question you are answering on each page;
 - No books, notes, computers, or calculators are allowed during the exam;
 - Please turn off cell-phones, iPhones, iPods, iPads, Kindles, and other electronic devices.

Grades

Q1.

Q2.

Q3.

Q4.

Q5.

Total

/70

QUESTION 1 [14 points]

A furnace contains a gas of N identical molecules in equilibrium at high temperature. Through a small window in the furnace one observes a spectral line of the gas molecules. The width of the observed line is broadened due to the Doppler effect. Derive the intensity $I(\lambda)$ as a function of the observed wavelength λ , the temperature T , the mass m of one molecule, the wavelength λ_0 of the spectral line when the molecule is at rest, the speed of light c , and N .

QUESTION 2 [14 points]

An electron in a magnetic field \vec{B} has energy $\vec{p}^2/2m \pm \kappa B$ according to whether the spin magnetic moment $\vec{\kappa}$ is parallel or anti-parallel to the magnetic field. (Here we set $B = |\vec{B}|$ and $\kappa = |\vec{\kappa}|$.) Calculate the paramagnetic susceptibility χ of a system of free electrons at very low temperatures when the electron gas is completely degenerate.

QUESTION 3 [14 points]

Consider a gas whose equation of state is given by the Van der Waals equation,

$$\left(P + \frac{a}{V^2}\right)(V - b) = NkT$$

with $a, b > 0$ constants.

- 7
3
4
- (a) Show that the heat capacity at constant volume C_V depends only on temperature T ;
 - (b) Calculate the internal energy E in terms of C_V as a function of T, V, N ;
 - (c) Next, suppose that C_V is constant, and that the gas is held in a container of negligible mass which is isolated from its surroundings. Initially, the gas is confined to 1/3 of the total volume of the container by a partition (a vacuum exists in the other 2/3), and is in equilibrium at temperature T_0 . Then, a hole is opened in the partition, allowing the gas to irreversibly expand and fill the entire volume V . What is the new temperature of the gas after thermal equilibrium is re-established ?

QUESTION 4 [14 points]

Consider an ideal gas of identical relativistic bosons, whose total number N is conserved. The relation between the energy ε and the momentum \vec{p} of a single boson is given by $\varepsilon = c|\vec{p}|$, where c is the speed of light.

- (a) Derive the condition for Bose-Einstein condensation in three space dimensions;
- (b) Derive a formula for the critical temperature;
- (c) Derive a formula for the fraction of the condensed bosons N_0 to their total number N as a function of temperature, and the other parameters of the problem.
- (d) Does Bose-Einstein condensation occur in two space dimensions ? Justify your answer.

QUESTION 5 [14 points]

Atoms in a solid vibrate about their respective equilibrium positions with small oscillations. Debye approximated the normal vibrations with the elastic vibrations of an isotropic body and assumed that the number of vibrational modes $g(\omega)d\omega$ having angular frequency between ω and $\omega + d\omega$ is given by,

$$g(\omega) = 9N \frac{\omega^2}{\omega_D^3} \theta(\omega_D - \omega) \quad (0.1)$$

Here, θ is the Heaviside step function, defined by $\theta(x) = 0$ for $x < 0$ and $\theta(x) = 1$ for $x > 0$, N is the number of atoms, and ω_D is the so-called Debye frequency (which is a constant whose precise value is dependent on the solid).

- 2 (a) Explain the choice of the normalization factor $9N$ in the function $g(\omega)$ given above.
- 4 (b) Derive the formulas in terms of $g(\omega)$ for the free energy F and the internal energy E .
- 4 (c) Calculate the specific heat at constant volume C_V of a solid with the Debye model.
[Do not attempt to evaluate any complicated integrals.]
- 4 (d) Determine the temperature dependence of C_V at high as well as at low temperatures, and sketch the behavior across all T .

STATISTICAL PHYSICS 215A

Final Exam – Spring 2014 – Solutions

QUESTION 1 [13 points]

- (a) **5 points** From the first law of thermodynamics, derive the following relation between the pressure P and the internal energy E , both of which are functions of T, V ,

$$\left(\frac{\partial E}{\partial V}\right)_T = T \left(\frac{\partial P}{\partial T}\right)_V - P$$

- (b) **4 points** The equation of state of a thermodynamics system is given by $P = \alpha \varepsilon(T)$ where α is a constant and $E = V \varepsilon(T)$. Calculate the temperature dependence of E .
- (c) **4 points** Calculate the entropy $S(T, V)$ for the system whose equation of state is given above.

Solution to Question 1

(a) At fixed N , and in terms of the independent variables T, V , the first law reads $dE(T, V) = TdS(T, V) - P(T, V)dV$. Identifying coefficients of dT and dV gives the relations,

$$\left.\frac{\partial E}{\partial T}\right|_V = T \left.\frac{\partial S}{\partial T}\right|_V \quad \left.\frac{\partial E}{\partial V}\right|_T = T \left.\frac{\partial S}{\partial V}\right|_T - P \quad (0.1)$$

Using now the Helmholtz free energy relation $dF = -SdT - PdV$, we have

$$\left.\frac{\partial P}{\partial T}\right|_V = \left.\frac{\partial S}{\partial V}\right|_T \quad (0.2)$$

The desired equation now readily follows by eliminating $\partial S/\partial V$ between the above equation and the second equation in (0.1).

(b) Using $P = \alpha \varepsilon(T)$ and $E = V \varepsilon(T)$ in the equation of (a), we find $(1 + \alpha)\varepsilon(T) = T\alpha\varepsilon'(T)$, which we integrate to,

$$\varepsilon(T) = \varepsilon_0 T^{(1+\alpha)/\alpha}$$

where ε_0 is a constant.

(c) Substituting the T -dependence of E into the first equation of (0.1), we find $T \frac{\partial S}{\partial T} = V \varepsilon'(T)$, so that the entropy is given by,

$$S(T, V) = S_0(V) + \varepsilon_0(1 + \alpha)VT^{1/\alpha}$$

Here, S_0 is a function of only V , but not T . In view of (0.2), however, it is found to be also independent of V , so it is just a constant, namely the zero temperature entropy (as long as $\alpha > 0$).

QUESTION 2 [14 points]

Consider a gas of N hard spheres in a box of volume V . A single sphere excludes a volume v around it. There are no interactions between the spheres, except for the constraints of the hard-core exclusion. The center of mass of a single sphere is free to move through the volume available to it and obeys the non-relativistic relation between energy, momentum and mass.

(a) 6 points

Calculate the entropy S , as a function of the total energy E ;

[Hint: use $\Gamma(x+1) = x\Gamma(x)$ and Sterling's formula $\ln \Gamma(x) = x \ln x - x + \mathcal{O}(\ln x)$.]

(b) 4 points

Calculate the equation of state for this gas;

(c) 4 points

Show that the isothermal compressibility $\kappa_T = -V^{-1}(\partial V / \partial P)|_T$ is always positive.

Solution to Question 2

(a) The number of micro-states available for fixed E, V, N is given by the expression,

$$\Omega(E, V, N) = \frac{1}{N!} \int_E \prod_{i=1}^N \frac{d^3 p_i d^3 q_i}{(2\pi\hbar)^3}$$

where the momenta are constrained to lie on the sphere in dimension $3N$ of radius square $2mE$. The evaluation of the integral over p_i is standard, and results in the expression,

$$\Omega(E, V, N) \approx \frac{1}{\Gamma(N)\Gamma(\frac{3N}{2})} \left(\frac{mE}{2\pi\hbar^2} \right)^{\frac{3N}{2}} \int \prod_{i=1}^N d^3 q_i$$

To compute the spatial volume, we produce by introducing each particle one by one into the box of volume V . The first particle has volume V available; the second $V - v$; the third $V - 2v$ and so on. All together, we obtain,

$$\int \prod_{i=1}^N d^3 q_i = V(V - v)(V - 2v) \cdots (V - (N - 1)v) = v^N \frac{\Gamma(\frac{V}{v} + 1)}{\Gamma(\frac{V}{v} - N + 1)}$$

Using Sterling's formula, we find,

$$\begin{aligned} S(E, V, N) &\approx \frac{3Nk}{2} \ln \left(\frac{mE}{3\pi\hbar^2 N} \right) + \frac{3Nk}{2} + \frac{V}{v} \ln \left(\frac{V}{V - Nv} \right) + N \ln \left(\frac{V - Nv}{N} \right) \\ &\approx \frac{3Nk}{2} \ln \left(\frac{mE}{3\pi\hbar^2 N} \right) + \frac{5Nk}{2} + Nk \ln \frac{V}{N} - \frac{N^2 kv}{2V} \end{aligned} \quad (0.3)$$

where we have used the approximation $Nv \ll V$ to retain only the leading correction in going from the first to the second line in the above formula.

(b) We use the micro-canonical formula for pressure, $P = T\partial S/\partial V$ at fixed E , to obtain,

$$P = \frac{NkT}{V - \frac{Nv}{2}}$$

Note that this formula holds true only to up to order $(vN/V)^2$ corrections.

(c) The isothermal compressibility is given by, $\kappa_T = NkT/(P^2V) > 0$.

QUESTION 3 [14 points]

We propose to evaluate the electric current density of electrons which is produced by heating up a metal in the presence of an external electric potential. The potential energy for electrons inside the metal vanishes, while just outside the metal it is $W > 0$. The electrons are considered otherwise non-interacting, and filled up to chemical potential μ with $\mu < W$. We consider the problem at sufficiently low temperature so that μ may be identified with the Fermi energy.

(a) 4 points

State the condition on its momentum for an electron to be able to escape from the metal to the outside, as a function of W and μ ;

(b) 6 points

Derive a general expression for the current density I of electrons leaving the metal;

(c) 4 points

Obtain an approximation of your result in (b) valid for sufficiently low temperatures.

Solution to Questions 3

(a) We take the edge of the metal where the electrons are being emitted to be orthogonal to the z -direction. The condition for an electron to be able to escape the metal to the outside is that its kinetic energy in the z -direction can overcome the potential energy outside the metal, $p_z^2/(2m) > W$ where m is the electron mass, and p_z is the electron momentum in the z -direction.

(b) The density of electrons inside the metal in an infinitesimal phase space volume $dV d^3p$ (where dV is the spacial volume element) is given by,

$$2 \frac{dV d^3p}{(2\pi\hbar)^3} \frac{1}{e^{\beta(\mathbf{p}^2/2m - \mu)} + 1} \quad (0.4)$$

The factor of 2 arises from the two spin states of the electron, and we have $\mathbf{p}^2 = p_x^2 + p_y^2 + p_z^2$. The electric current density is then given by the thermal expectation value of the observable, $e p_z/m$ per unit volume, restricted to the range $p_z > \sqrt{2mW}$. Thus the current density $I = I_z$ is given by the following integral,

$$I_z = 2 \frac{e}{m} \frac{1}{(2\pi\hbar)^3} \int_{\sqrt{2mW}}^{\infty} dp_z \int_{-\infty}^{\infty} dp_x \int_{-\infty}^{\infty} dp_y \frac{p_z}{e^{\beta(\mathbf{p}^2/2m - \mu)} + 1} \quad (0.5)$$

Changing variables to the following dimensionless combinations s, t defined by,

$$s = \beta \left(\frac{p_z^2}{2m} - W \right) \quad t = \beta \frac{p_x^2 + p_y^2}{2m} \quad (0.6)$$

The integral for I_z reduces to,

$$I_z = \frac{e m}{2\pi^2 \hbar^3} (kT)^2 \int_0^{\infty} ds \int_0^{\infty} dt \frac{1}{e^{s+t+\beta(W-\mu)} + 1} \quad (0.7)$$

(c) For sufficiently low temperatures, namely $T \ll W - \mu$, we may drop the 1 in the denominator, and carry out the integrals over s and t explicitly. We are then left with the following approximate formula,

$$I_z = \frac{e m}{2\pi^2 \hbar^3} (kT)^2 \exp \left\{ -\frac{W - \mu}{kT} \right\} \quad (0.8)$$

The exponential behavior is consistent with the existence of an energy gap of size $W - \mu$.

QUESTION 4 [14 points]

(a) 3 pts

For any system of fermions at chemical potential μ and temperature T , show that the probability for finding an occupied state of energy $\varepsilon + \mu$ is the same as that for finding an unoccupied state of energy $\mu - \varepsilon$.

Consider now a system of non-interacting Dirac fermions of spin 1/2 and mass m . One-particle states at momentum \mathbf{p} come in pairs of positive and negative energy,

$$\varepsilon_{\pm}(\mathbf{p}) = \pm \sqrt{m^2 c^4 + \mathbf{p}^2 c^2}$$

for each value of the spin quantum number. At $T = 0$, all negative energy Dirac states are filled (the so-called Dirac-sea), and all positive energy states are empty, so that $\mu(T = 0) = 0$.

- (b) **3 pts** Using the result of (a) compute the chemical potential at arbitrary temperature T .
 (c) **3 pts** Compute (an integral representation for) the mean excitation energy $E(T) - E(0)$.
 (d) **2 pts** Evaluate the integral in part (c) for $m = 0$ and evaluate the specific heat C_V ;
 (e) **3 pts** Describe the qualitative change in the specific heat at low temperature when $m \neq 0$.

Solution to Questions 4

(a) The probabilities for finding occupied states at energy $\mu + \varepsilon$ and energy $\mu - \varepsilon$ are respectively given by,

$$n(\mu + \varepsilon) = \frac{1}{e^{\beta\varepsilon} + 1} \qquad n(\mu - \varepsilon) = \frac{1}{e^{-\beta\varepsilon} + 1}$$

which are clearly related by $n(\mu - \varepsilon) = 1 - n(\mu + \varepsilon)$, and the right side is precisely the probability for finding the state unoccupied at energy $\mu + \varepsilon$. QED.

(b) Total particle number is unchanged at finite temperature, and using the above particle-hole symmetry, the chemical potential remains zero at any temperature, $\mu(T) = 0$.

(c) Formally, the internal energy is given by summing over the contributions from positive and negative energy states, with their associated occupation numbers,

$$E(T) = 2V \int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} \left(\varepsilon_+(\mathbf{p}) n(\mu + \varepsilon_+(\mathbf{p})) + \varepsilon_-(\mathbf{p}) n(\mu + \varepsilon_-(\mathbf{p})) \right)$$

where the prefactor of 2 arises because of the electron spin degeneracy. We now use $\varepsilon_-(\mathbf{p}) = -\varepsilon_+(\mathbf{p})$ as well as the result of (a), namely $n(\mu + \varepsilon_-(\mathbf{p})) = n(\mu - \varepsilon_+(\mathbf{p})) = 1 - n(\mu + \varepsilon_+(\mathbf{p}))$. Hence, we have

$$E(T) = 2V \int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} \left(2\varepsilon_+(\mathbf{p}) n(\mu + \varepsilon_+(\mathbf{p})) + \varepsilon_-(\mathbf{p}) \right)$$

Omitting the T -independent contribution of the term in $\varepsilon_-(\mathbf{p})$ in the integrand amounts to subtracting the energy of the negative energy Dirac sea. Also, setting now $\mu = 0$ by the result of (b), we find,

$$E(T) - E(0) = 4V \int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} \frac{\varepsilon_+(\mathbf{p})}{e^{\beta\varepsilon_+(\mathbf{p})} + 1}$$

The multiplicity is explained as follows. One factor of 2 arises from spin 1/2, while the other arises from the contributions of both positive and negative energy states.

(d) For $\varepsilon_+(\mathbf{p}) = c|\mathbf{p}|$, we have,

$$E(T) - E(0) = 4V \int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} \frac{c|\mathbf{p}|}{e^{\beta c|\mathbf{p}|} + 1} = \frac{2V}{\pi^2\hbar^3 c^3} (kT)^4 \int_0^\infty dx \frac{x^3}{e^x + 1}$$

The value of the last integral is $6\zeta(4) = \pi^4/15$, but it is not essential that it be evaluated.

(e) When $m \neq 0$, the energy spectrum develops a gap of size $2mc^2$, so that there must be a suppression factor $\exp\{-mc^2/kT\}$ at low temperatures.

MIDTERM 1 SOLUTIONS

Started: May 6, 2014
Last updated: May 7, 2014

Question 1.

Set the energy of a crystal with no defects to be zero, then it's energy with n defects is $E = wn$. It follows that the temperature of the system satisfies

$$\frac{1}{T} = \frac{\partial S}{\partial E} = \frac{\partial S}{\partial n} = \frac{\partial S}{\partial n} \frac{\partial n}{\partial E} = \frac{1}{w} \frac{\partial S}{\partial n}. \quad (1)$$

Now we simply evaluate $\partial S/\partial n$ in the regime $1 \ll n \ll N$. The number of microstates corresponding to n defects out of a total of N sites is

$$\Omega(n, N) = \binom{N}{n} = \frac{N!}{n!(N-n)!} \quad (2)$$

So, by using Stirling's approximation $\ln k! \approx k \ln k - k$, we find

$$\begin{aligned} \frac{S(n, N)}{k} &= \ln \Omega(n, N) \\ &= \ln N! - \ln(N-n)! - \ln n! \\ &\approx N \ln N - N - (N-n) \ln(N-n) + (N-n) - n \ln n + n \\ &= N \ln N - (N-n) \ln(N-n) - n \ln n \end{aligned} \quad (3)$$

and therefore

$$\frac{\partial S}{\partial n} \approx k \ln \frac{N-n}{n} \quad (4)$$

Combining this with (1) gives

$$\boxed{n \approx N e^{-w/kT}} \quad (5)$$

Question 2.

(a) Using the notation $p = |\mathbf{p}|$, the partition function of the system (assuming indistinguishable particles) is

$$Z = \frac{1}{N!} \int \frac{d^{3N}p d^{3N}q}{h^{3N}} e^{-\beta H(p,q)} = \frac{1}{N!} \left[\int \frac{d^3p d^3q}{h^3} e^{-\beta cp} \right]^N = \frac{(4\pi V)^N}{N! h^{3N}} \left[\int_0^\infty dp p^2 e^{-\beta cp} \right]^N \quad (6)$$

we can convert the integral in brackets to a dimensionless integral by making the substitution $u = \beta cp$, then

$$Z = \frac{(4\pi V)^N}{N! h^{3N}} \frac{I^N}{(\beta c)^{3N}}, \quad I \equiv \int_0^\infty du u^2 e^{-u} = 2 \quad (7)$$

(b) The simplest way to compute the ensemble average energy E is to take an appropriate derivative of the partition function;

$$E = -\frac{\partial \ln Z}{\partial \beta} = -\frac{1}{Z} \frac{\partial Z}{\partial \beta} = -\frac{1}{Z} (-3N\beta^{-1} Z) = \boxed{3NkT}. \quad (8)$$

(c) The density of states is defined as;

$$Z(\beta) = \int_0^\infty dE e^{-\beta E} g(E). \quad (9)$$

Namely, the partition function is the Laplace transform of the density of states. To compute the density of states, we need to invert this transform. In general, inverting a Laplace transform may not be so easy, but in this problem, we can be clever. The partition function Z is of the form $\nu \beta^{-3N}$, so density of states satisfies

$$\nu \beta^{-3N} = \int_0^\infty dE e^{-\beta E} g(E). \quad (10)$$

The trick is to change variables $u = \beta E$ so that

$$\frac{\nu}{\beta^{3N-1}} = \int_0^\infty du e^{-u} g\left(\frac{u}{\beta}\right) \quad (11)$$

We can now see by inspection (think of performing integration by parts $3N - 1$ times), that this equation is solved by a function g of the form

$$g(E) = \alpha E^{3N-1} \quad (12)$$

We can determine α by plugging it back into the equation;

$$\alpha = \nu \left(\int_0^\infty du e^{-u} u^{3N-1} \right)^{-1} = \frac{\nu}{(3N-1)!} \quad (13)$$

Putting this all together, and noting the value of ν from part (a) of this problem in terms of the other given parameters, we find that

$$g(E) = \frac{(4\pi V)^N}{N! h^{3N}} \frac{2^N}{c^{3N}} \frac{E^{3N-1}}{(3N-1)!} \quad (14)$$

Question 3.

Method 1.

We appeal to our knowledge from problem 4.2 on the homework. We note that determining the number density $n = \rho/m$, where m is the mass of each molecule, amounts to finding a critical point of the Helmholtz free energy functional. We go into the rotating frame of the cylinder in which case the energy of the gas is $E[n] = E_0 + E_{\text{cf}}[n]$ where E_0 is the energy that gas would have if the cylinder were non-rotating, and E_{cf} is the extra energy due to the centrifugal force. If we appeal to rotational symmetry to write the density as only a function of the cylindrical coordinate r , then

$$E_{\text{cf}}[n] = - \int_{\text{cyl.}} \frac{1}{2} dm(r) r^2 \omega^2 = -\frac{1}{2} m \omega^2 (2\pi L) \int_0^R dr r^3 n(r) \quad (15)$$

In addition, we assume, as in the homework, that the entropy of the system for a given number density n can be written as

$$S[n] = -k \int_{\text{cyl.}} d^3\mathbf{x} n(\mathbf{x}) \ln n(\mathbf{x}) = -k(2\pi L) \int_0^R dr r n(r) \ln n(r) \quad (16)$$

where we have appealed to rotational symmetry to write the density purely as a function of the cylindrical radial coordinate r . We want to determine the density n satisfying $\delta F[n] = 0$ for all variations of n that keep the total particle number $N = \int n$ fixed. The Helmholtz free energy is $E[n] - TS[n]$, so using (15) and (16), we find that the condition for finding a critical point of the free energy is

$$0 = \int_0^R dr r \left(kT \ln n(r) + kT - \frac{1}{2} m \omega^2 r^2 \right) \delta n(r) \quad (17)$$

Now, recall that we have the total particle number constraint, which tells us that only variations satisfying the following constraint are admissible:

$$\int_0^R dr r \delta n(r) = 0 \quad (18)$$

Noting that (17) holds for all variations satisfying (18), we find that there exists a constant w for which

$$kT \ln n(r) - \frac{1}{2} m \omega^2 r^2 = w. \quad (19)$$

It follows that

$$n(r) = n(0) \exp \left(\frac{m \omega^2 r^2}{2kT} \right) \quad (20)$$

Plugging this back into the particle number constraint $N = \int n$ allows us to compute the normalization $n(0)$. After normalizing and noting that $\rho(r) = mn(r)$, we find that

$$\rho(r) = \frac{Nm}{\pi R^2 L} \frac{m\omega^2 R^2}{2kT} \left[\exp\left(\frac{m\omega^2 R^2}{2kT}\right) - 1 \right]^{-1} \exp\left(\frac{m\omega^2 r^2}{2kT}\right) \quad (21)$$

Method 2.

Note that the mass density ρ can be written as a function on the $3N + 3N$ dimensional phase space of the system as follows:

$$\rho(\mathbf{x}; p, q) = m \sum_{i=1}^N \delta^{(3)}(\mathbf{x} - \mathbf{x}_i), \quad (p, q) \equiv (\mathbf{p}_1, \dots, \mathbf{p}_N, \mathbf{x}_1, \dots, \mathbf{x}_N) \quad (22)$$

One can then determine the ensemble average density as follows

$$\langle \rho(\mathbf{x}) \rangle = \frac{1}{Z} \int \frac{d^{3N}p d^{3N}x}{h^{3N}} e^{-\beta H_*(p, q)} \rho(\mathbf{x}; p, q), \quad H_* = H - \omega L \quad (23)$$

and this reproduces the answer that employs the first method.

STATISTICAL PHYSICS 215A

Final Exam – Spring 2013 – SOLUTIONS

1 Solution to Question 1

(a) The specific heats in question are defined by

$$C_M = \left(\frac{\partial E}{\partial T} \right)_M \quad C_H = \left(\frac{\partial(E - MH)}{\partial T} \right)_H \quad (1.1)$$

(b) In the canonical ensemble, the independent variables are (T, M) , and we have $E(T, M)$. To compute C_H , we use independent variables T, H , so that M is a function of T, H which we denote $M(T, H)$. Thus, we have

$$\begin{aligned} C_H &= \left. \frac{\partial(E - HM)}{\partial T} \right|_H = \left. \frac{\partial E}{\partial T} \right|_H - H \left. \frac{\partial M}{\partial T} \right|_H \\ \left. \frac{\partial E}{\partial T} \right|_H &= \left. \frac{\partial E(T, M(T, H))}{\partial T} \right|_H = \left. \frac{\partial E}{\partial T} \right|_M + \left. \frac{\partial E}{\partial M} \right|_T \left. \frac{\partial M}{\partial T} \right|_H \end{aligned} \quad (1.2)$$

from which the expression of (a) follows immediately.

(c) Curie's law allows us to express H in terms of (T, M) , and we have,

$$dE = TdS + \frac{MT}{nD}dM \quad (1.3)$$

Since C_M is constant, and $\partial E/\partial M = 0$ at constant T , we have $dE = C_M dT$. During adiabatic transformations, we have $dS = 0$, so that we obtain the differential equation,

$$C_M dT = \frac{MT}{nD}dM \quad (1.4)$$

Dividing both sides by T separates the variables, and allows us to integrate by quadrature,

$$T(M) = T(M=0) \exp \left\{ \frac{M^2}{2nDC_M} \right\} \quad (1.5)$$

(d) The transformations $2 \rightarrow 3$ and $4 \rightarrow 1$ are adiabatic; using (1.5), we have,

$$\begin{aligned} M_3^2 &= M_2^2 - 2nDC_M \ln \frac{T_h}{T_\ell} \\ M_4^2 &= M_1^2 - 2nDC_M \ln \frac{T_h}{T_\ell} \end{aligned} \quad (1.6)$$

Note that the signs work out as $T_h > T_\ell$, while we have $M_3^2 < M_2^2$ and $M_4^2 < M_1^2$.

The heat liberated along an isothermal may be computed from the fact that for fixed T the internal energy E remains constant for this specific material in view of the assumption that $\partial E/\partial M = 0$ at constant T ,

$$0 = \delta Q + H dM = \delta Q + \frac{MT}{nD} dM \quad (1.7)$$

The heat liberated along the isothermal processes $1 \rightarrow 2$ and $3 \rightarrow 4$ is respectively given by,

$$\begin{aligned} Q_h &= -\frac{T_h}{2nD}(M_1^2 - M_2^2) \\ Q_\ell &= -\frac{T_\ell}{2nD}(M_4^2 - M_3^2) \end{aligned} \quad (1.8)$$

As internal energy is conserved along the isothermals, we have

$$W_{1 \rightarrow 2} = Q_h \quad W_{3 \rightarrow 4} = Q_\ell \quad (1.9)$$

while along the adiabatics, the work equals the change in internal energy, and we have,

$$W_{2 \rightarrow 3} = -C_M(T_h - T_\ell) \quad W_{4 \rightarrow 1} = -C_M(T_\ell - T_h) \quad (1.10)$$

Elimination T_h/T_ℓ between the two equations in (1.6), we find $M_1^2 - M_2^2 = M_4^2 - M_3^2$, so that we find $Q_h/Q_\ell = T_h/T_\ell$. The total work done by the system is given by conservation of total internal energy by $W = Q_h - Q_\ell$. By definition of η , we have,

$$\eta = \frac{W}{Q_h} = 1 - \frac{Q_\ell}{Q_h} = 1 - \frac{T_\ell}{T_h} \quad (1.11)$$

Note that by the second law of thermodynamics, W, Q_h , and Q_ℓ are all positive.

2 Solution to Question 2

(a) The total number of micro-states $\Omega(E, L, N)$ is given by the multiple integral,

$$\Omega(E, L, N) = \frac{1}{N!} \prod_{i=1}^N \int \frac{dq_i dp_i}{2\pi\hbar} \theta \left(E - c \sum_{i=1}^N |p_i| \right) \quad (2.1)$$

where θ denotes the Heaviside step function. The factor of $1/N!$ is included to account for the indistinguishability of the particles stated in the problem. The range of each integral in q_i is over the box of length L . By symmetry of the integrand under $p_i \rightarrow -p_i$, we restrict

the range of integration in p_i to $[0, \infty]$, and include a factor of 2 for each integration. Thus, we end up with the simplified formula,

$$\begin{aligned}\Omega(E, L, N) &= \frac{1}{N!} \left(\frac{2L}{2\pi\hbar} \right)^N \mathcal{V}(N, E/c) \\ \mathcal{V}(N, \lambda) &= \prod_{i=1}^N \int_0^\infty dp_i \theta \left(\lambda - \sum_{i=1}^N |p_i| \right)\end{aligned}\tag{2.2}$$

By scaling all p_i , we see that we have,

$$\mathcal{V}(N, \lambda) = \lambda^N \mathcal{V}(N, 1)\tag{2.3}$$

On the other hand, the integral may be evaluated iteratively,

$$\begin{aligned}\mathcal{V}(N, \lambda) &= \int_0^\lambda dp_N \mathcal{V}(N-1, \lambda - p_N) \\ &= \int_0^\lambda dp_N (\lambda - p_N)^{N-1} \mathcal{V}(N-1, 1) = \frac{\lambda^N}{N} \mathcal{V}(N-1, 1)\end{aligned}\tag{2.4}$$

Putting all together, we have $\mathcal{V}(N, \lambda) = \lambda^N/N!$ so that,

$$\Omega(E, L, N) = \frac{1}{(N!)^2} \left(\frac{LE}{\pi\hbar c} \right)^N\tag{2.5}$$

It immediately follows that

$$\Omega'(E, L, N, \Delta) = \frac{1}{(N!)^2} \left(\frac{LE}{\pi\hbar c} \right)^N \frac{N\Delta}{E}\tag{2.6}$$

(b) The entropy is defined in terms of the number of micro-states at energy E , so it should be in terms of Ω' . We shall adopt the following definition,

$$S(E, L, N) = k \ln \frac{\Omega'(E, L, N, \Delta)}{\Delta} = k \ln \left(\frac{1}{(N!)^2} \left(\frac{LE}{\pi\hbar c} \right)^N \frac{N}{E} \right)\tag{2.7}$$

In the thermodynamic limit, the contribution from the factor N/E cancels out, and we omit it outright. The remaining expression in the limit may be rearranged as follows,

$$\frac{S}{N} = k \ln \left(\frac{LE}{\pi\hbar c N^2} \right) + 2k\tag{2.8}$$

We have used Sterling's formula to obtain the last term. The entropy is properly extensive.

(c) Using the micro-canonical ensemble, we have from the definition of temperature,

$$\frac{1}{T} = \left. \frac{\partial S}{\partial E} \right|_{N,L} = \frac{kN}{E} \quad (2.9)$$

The resulting relation $E = NkT$ violates the equipartition theorem by a factor of 2. In the canonical ensemble, we calculate the partition function,

$$Z = \frac{1}{N!} \prod_{i=1}^N \left(\int \frac{dp_i dq_i}{2\pi\hbar} e^{-\beta c|p_i|} \right) = \frac{1}{N!} \left(\frac{kTL}{\pi\hbar c} \right)^N \quad (2.10)$$

The internal energy is deduced directly from

$$E = -\frac{\partial \ln Z}{\partial \beta} = NkT \quad (2.11)$$

which agrees with the earlier calculation in the micro-canonical ensemble.

(d) Returning to the micro-canonical ensemble, the pressure is given by,

$$\frac{P}{T} = \left. \frac{\partial S}{\partial L} \right|_{E,N} = \frac{Nk}{L} \quad (2.12)$$

so that $P = NkT/L$, while we have,

$$C_L = \left. \frac{\partial E}{\partial T} \right|_{L,N} = kN \quad (2.13)$$

3 Solution to Question 3

(a) In terms of the density of one-particle states $D(\varepsilon)$ and the Fermi occupation number f , the total number of particles N and the internal energy E are given by,

$$\begin{aligned} N &= V \int_0^\infty d\varepsilon D(\varepsilon) f(\varepsilon) \\ E &= V \int_0^\infty d\varepsilon \varepsilon D(\varepsilon) f(\varepsilon) \end{aligned} \quad (3.1)$$

We assume that $D(\varepsilon) = 0$ for $\varepsilon < 0$. Since the function $D(\varepsilon)$ does not involve temperature (but may involve V which is held constant in computing C_V), the specific heat is given by,

$$C_V = \frac{\partial E}{\partial T} = V \int_0^\infty d\varepsilon \varepsilon D(\varepsilon) \frac{\partial f(\varepsilon)}{\partial T} \quad (3.2)$$

Note that since we are using the grand-canonical ensemble here, this specific heat corresponds to holding μ fixed (instead of in the canonical ensemble where we would hold N fixed instead). Working this out, and after some minor simplifications, we get,

$$C_V = \frac{V}{kT^2} \int_0^\infty d\varepsilon \frac{\varepsilon(\varepsilon - \mu)D(\varepsilon)}{(e^{\beta(\varepsilon-\mu)/2} + e^{-\beta(\varepsilon-\mu)/2})^2} \quad (3.3)$$

(b) Strong degeneracy corresponds to low temperatures. The denominator is then responsible for concentrating the support of the integral over ε near μ , so we may extend the integration region all the way to $-\infty$. Also, to leading order, we may evaluate $D(\varepsilon)$ at the central value μ . The parity of the remaining integral allows us to replace the factor ε in the numerator by $\varepsilon - \mu$, so that we end up with the following expression,

$$C_V = \frac{VD(\mu)}{kT^2} \int_{-\infty}^\infty d\varepsilon \frac{(\varepsilon - \mu)^2}{(e^{\beta(\varepsilon-\mu)/2} + e^{-\beta(\varepsilon-\mu)/2})^2} \quad (3.4)$$

Changing variables from ε to x with $\varepsilon = \mu + 2kTx$ gives,

$$C_V = 8k^2TV D(\mu) \int_{-\infty}^\infty dx \frac{x^2}{(e^x + e^{-x})^2} \quad (3.5)$$

Using the value of the integral stated in the problem, and the fact that for small temperatures we have $\mu = \mu_0 + \mathcal{O}(T^2)$, with μ_0 defined by,

$$N = V \int_0^{\mu_0} d\varepsilon D(\varepsilon) \quad (3.6)$$

we approximate this result by setting $D(\mu) = D(\mu_0)$, so that the final result is given by,

$$C_V = \frac{1}{3}\pi^2 k^2 V D(\mu_0) T \quad (3.7)$$

Observe from equation (3.6) that holding μ fixed is equivalent to holding N fixed in the approximation of $T = 0$; thus C_V computed at fixed μ as was done here in the grand-canonical ensemble coincides with C_V computed at fixed N .

(c) Weak degeneracy corresponds to high temperature T , in which case the FD occupation number becomes the Boltzmann number, and we have,

$$C_V = \frac{V}{kT^2} \int_0^\infty d\varepsilon \varepsilon(\varepsilon - \mu)D(\varepsilon) e^{-\beta(\varepsilon-\mu)} \quad (3.8)$$

with μ obtained from,

$$N = V \int_0^\infty d\varepsilon D(\varepsilon) e^{-\beta(\varepsilon-\mu)} \quad (3.9)$$

In both the formulas for C_V and N , the fugacity $e^{\beta\mu}$ factors out from under the integrations, and we may eliminate it entirely, to obtain the formula,

$$C_V = \frac{N}{kT^2} \frac{\int_0^\infty d\varepsilon \varepsilon (\varepsilon - \mu) D(\varepsilon) e^{-\beta\varepsilon}}{\int_0^\infty d\varepsilon D(\varepsilon) e^{-\beta\varepsilon}} \quad (3.10)$$

Note that, as earlier, this is still the specific heat at constant μ , not constant N .

(d) The density of states for a non-relativistic free particle is given by

$$g \int \frac{d^3p d^3q}{(2\pi\hbar)^3} f(\varepsilon) = p^2/2m = \frac{2\pi g V (2m)^{3/2}}{(2\pi\hbar)^3} \int_0^\infty d\varepsilon \sqrt{\varepsilon} f(\varepsilon) \quad (3.11)$$

where $g = 2$ for the electron. Thus, we conclude that

$$D(\varepsilon) = \frac{2\pi g V (2m)^{3/2}}{(2\pi\hbar)^3} \sqrt{\varepsilon} \quad (3.12)$$

For low T , formula (3.6) gives μ_0 as a function of the number density,

$$\frac{N}{V} = \frac{4\pi g}{3} \left(\frac{2m\mu_0}{(2\pi\hbar)^2} \right)^{3/2} \quad (3.13)$$

so that the specific heat takes the form,

$$\frac{C_V}{Nk} = \frac{mkT}{4\hbar^2} \left(\frac{4\pi g V}{3N} \right)^{2/3} \quad (3.14)$$

For high T , the specific heat at constant μ is given by

$$\begin{aligned} \frac{C_V}{Nk} &= \beta^2 \frac{\int_0^\infty d\varepsilon \varepsilon^{3/2} (\varepsilon - \mu) e^{-\beta\varepsilon}}{\int_0^\infty d\varepsilon \varepsilon^{1/2} e^{-\beta\varepsilon}} \\ &= \frac{\int_0^\infty dx x^{3/2} (x - \beta\mu) e^{-x}}{\int_0^\infty dx x^{1/2} e^{-x}} \\ &= \frac{\Gamma(7/2)}{\Gamma(3/2)} - \frac{\Gamma(5/2)}{\Gamma(3/2)} \beta\mu = \frac{15}{4} - \frac{3}{2} \beta\mu \end{aligned} \quad (3.15)$$

At high T the last term may be neglected. We do not find the standard $3/2$ because this is not the specific heat at constant N . To obtain the latter in the large T limit, one should first eliminate μ in favor of N , so that one first obtains,

$$\frac{E}{N} = \frac{\int_0^\infty d\varepsilon \varepsilon D(\varepsilon) e^{-\beta\varepsilon}}{\int_0^\infty d\varepsilon D(\varepsilon) e^{-\beta\varepsilon}} = \frac{3}{2} kT \quad (3.16)$$

The standard result $C_V = 3Nk/2$ then follows.

4 Solution to Question 4

(a) The Hamiltonian for this system depends on the height ℓ of the matter in the cylinder of total height L , with $0 < \ell < L$, and is given by,

$$H = \frac{\mathbf{p}^2}{2m} + mg\ell \quad (4.1)$$

The corresponding number density is given by,

$$N - N_0 = \frac{V}{L} \int_0^L d\ell \int \frac{d^3p}{(2\pi\hbar)^3} \frac{1}{e^{\beta(\mathbf{p}^2/2m + mg\ell - \mu)} - 1} \quad (4.2)$$

The expression may be recast in terms of the function $g_{3/2}$. Setting $\ell = Ly$, we obtain,

$$N - N_0 = \frac{V}{\lambda^3} \int_0^1 dy g_{3/2} \left(z e^{-\beta mgLy} \right) \quad (4.3)$$

Criticality in the absence of gravity is at temperature T_c^0 , with corresponding value λ_0 , and is determined by setting $z = 1$, so that we have,

$$N = \frac{V}{\lambda_0^3} g_{3/2}(1) = \frac{V}{\lambda_0^3} \zeta(3/2) \quad (4.4)$$

Criticality in the presence of gravity at temperature T_c with corresponding value λ is determined by setting $z = 1$, so that we have the relation,

$$N = \frac{V}{\lambda^3} \int_0^1 dy g_{3/2} \left(e^{-\beta mgLy} \right) \quad (4.5)$$

Since we have $mgL \ll kT_c$, we may uniformly expand the integrand as follows,

$$g_{3/2} \left(e^{-\beta mgLy} \right) = \zeta(3/2) - 2\sqrt{\pi}(\beta mgLy)^{\frac{1}{2}} + \mathcal{O}(\beta mgL) \quad (4.6)$$

Combining equations (4.4) with (4.5) in the above approximation, we find,

$$\frac{1}{\lambda^3} = \frac{1}{\lambda_0^3} \int_0^1 dy \left(1 - \frac{2\sqrt{\pi}}{\zeta(3/2)} (\beta mgLy)^{\frac{1}{2}} + \mathcal{O}(\beta mgLy) \right) \quad (4.7)$$

Within the approximation of small gravitational effects, we have,

$$T_c = T_c^0 \left(1 + \frac{8\sqrt{\pi}}{9\zeta(3/2)} \left(\frac{mgL}{kT_c^0} \right)^{\frac{1}{2}} + \mathcal{O}(g) \right) \quad (4.8)$$

(b) In the condensed phase, we set $z = 1$, so that the internal energy is given by,

$$\frac{E}{V} = \frac{3kT}{2\lambda^3} \int_0^1 dy g_{5/2} \left(e^{-\beta mgLy} \right) \quad (4.9)$$

From the asymptotics of $g_{3/2}$, we deduce the asymptotics of $g_{5/2}$, and we find,

$$g_{5/2}(e^{-\alpha}) = \zeta(5/2) - \zeta(3/2)\alpha + \frac{4}{3}\sqrt{\pi}\alpha^{3/2} + \mathcal{O}(\alpha^2) \quad (4.10)$$

From this expansion, it is clear that the next order correction arising from the integral is suppressed by a factor of mgL/kT_c , so this is subleading compared to the square root behavior. As a result, we find,

$$\frac{E}{V} = \frac{3kT}{2\lambda^3} \zeta(5/2) \quad (4.11)$$

so that the specific heat is given by,

$$\frac{C_V}{Nk} = \frac{15}{4} \frac{\zeta(5/2)}{\zeta(3/2)} \frac{\lambda_0^3}{\lambda^3} \quad (4.12)$$

where λ corresponds to T_c , while λ_0 corresponds to T_c^0 .

In the gas phase, we have instead,

$$\frac{E}{V} = \frac{3kT}{2\lambda^3} \int_0^1 dy g_{5/2} \left(ze^{-\beta mgLy} \right) \quad (4.13)$$

Since $g_{5/2}(z)$ has a regular expansion near $z = 1$ to leading non-trivial order, the exponential correction in the argument may be neglected, as it is of higher order. Thus we are left with,

$$\frac{E}{V} = \frac{3kT}{2\lambda^3} g_{5/2}(z) \quad (4.14)$$

The specific heat is obtained by differentiating with respect to T , including the dependence on z ,

$$\frac{C_V}{Nk} = \frac{15}{4} \frac{g_{5/2}(z)}{g_{3/2}(z)} \frac{\lambda_0^3}{\lambda^3} + \frac{3}{2} \frac{\lambda_0^3}{\lambda^3} \frac{\partial \ln z}{\partial \ln T} \Big|_{N,V} \quad (4.15)$$

The remaining derivative is obtained from differentiating the relation

$$\frac{N}{V} \lambda^3 = \int_0^1 dy g_{3/2} \left(ze^{-\beta mgLy} \right) \quad (4.16)$$

with respect to T , while keeping N and V fixed. Setting $z = 1$ to match with the condensed phase, and again neglecting higher orders in mgL/kT_c , we find the discontinuity to be given by,

$$\frac{\Delta C_V}{Nk} = -\frac{9}{4} \frac{\zeta(3/2)}{\int_0^1 dy g_{1/2}(e^{-\beta mgLy})} \quad (4.17)$$

Using the expansion of $g_{1/2}$ near 1, derived from that of $g_{3/2}$, we find,

$$g_{1/2}(e^{-\alpha}) = \sqrt{\frac{\pi}{\alpha}} + \mathcal{O}(1) \quad (4.18)$$

so that

$$\frac{\Delta C_V}{Nk} = -\frac{9\zeta(3/2)}{8\sqrt{\pi}} \left(\frac{mgL}{kT_c^0} \right)^{\frac{1}{2}} \quad (4.19)$$

(1)

Statistical Physics 215 A.

Final Exam - Spring 2012

Solutions.

1. Due to the Doppler effect the wavelength of emission of an atom with velocity v_x with respect to the observer is given by

$$\lambda \cong \lambda_0 \left(1 + \frac{v_x}{c}\right) \quad v_x/c \ll 1.$$

The velocity v_x is governed by the Maxwell distribution of the atoms inside the furnace. The Maxwell distribution is normalized by the canonical partition function for free particles,

$$Z = \int \frac{d^3p V}{(2\pi\hbar)^3} e^{-\vec{p}^2/2mkT} = \frac{m^3 V}{(2\pi\hbar)^3} \left(\frac{2\pi kT}{m}\right)^{3/2}$$

The probability distribution for the velocity is given by

$$P_{\text{max}} = \frac{d^3p}{m^3} \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\vec{v}^2 \frac{m}{2kT}}$$

$$d^3v$$

Since we are interested only in the distribution for v_x , we integrate over v_y, v_z :

(2)

$$p(v_x) = dv_x \left(\frac{m}{2\pi kT} \right)^{\frac{1}{2}} e^{-\frac{m}{2kT} v_x^2}$$

Expressing $v_x = c \frac{\lambda - \lambda_0}{\lambda_0}$, we find

$$p(\lambda) = \frac{c}{\lambda_0} d\lambda \left(\frac{m}{2\pi kT} \right)^{\frac{1}{2}} e^{-\frac{mc^2}{2kT\lambda_0^2} (\lambda - \lambda_0)^2}$$

The intensity is $p(\lambda) N$.

(3)

2. The energy levels of the electron are

$$\varepsilon = \frac{\vec{p}^2}{2m} \pm KB$$

where \pm refers to the two eigenvalues of spin along B.

At $T=0$, the electrons occupy all levels up to a chemical potential μ_0 . Thus, the kinetic energies of the \pm spin states range over:

$$0 < |\vec{p}| < p_+ \quad \frac{1}{2m} p_+^2 + KB = \mu_0.$$

$$0 < |\vec{p}| < p_- \quad \frac{1}{2m} p_-^2 - KB = \mu_0.$$

The number of spins in each state are

$$N_+ = \frac{4\pi V}{3(2\pi\hbar)^3} p_+^3$$

$$N_- = \frac{4\pi V}{3(2\pi\hbar)^3} p_-^3$$

The total magnetization is thus,

$$M = -K(N_+ - N_-) = -\frac{4\pi V}{(2\pi\hbar)^3} K(p_+^3 - p_-^3).$$

$$M = -\frac{4\pi V}{(2\pi\hbar)^3} K \left[(2m(\mu_0 - KB))^{\frac{3}{2}} - (2m(\mu_0 + KB))^{\frac{3}{2}} \right]$$

(4)

Assuming small magnetic fields $kB \ll \mu_0$,

$$(\mu_0 \pm kB)^{\frac{3}{2}} = \mu_0^{\frac{3}{2}} \left(1 \pm \frac{3}{2} k \frac{B}{\mu_0} \right)$$

Hence

$$M = \frac{4\pi V}{(2\pi\hbar)^3} k (2m\mu_0)^{\frac{3}{2}} 3k \frac{B}{\mu_0} + O(B^3).$$

and the total number of electrons is:

$$N = N_+ + N_- = \frac{4\pi V}{(2\pi\hbar)^3} 2(2m\mu_0)^{\frac{3}{2}}$$

Combining both:

$$M = \frac{3}{2} N \frac{k^2}{\mu_0} B$$

Hence the molecular susceptibility is given by

$$\chi = \frac{1}{N} \frac{\partial M}{\partial B} = \frac{3}{2} \frac{k^2}{\mu_0}$$

(5)

3. Specific heat at constant V is given by

$$C_V = \left. \frac{\partial E}{\partial T} \right|_{V, N}$$

(a) We have available P , and this allows us to compute

$$P = - \left. \frac{\partial F}{\partial V} \right|_{T, N}$$

from Van der Waals:

$$P = - \frac{a}{V^2} + \frac{NkT}{V-b}$$

$$F = - \frac{a}{V} - NkT \ln(V-b) + F_0(T)$$

where $F_0(T)$ depends only on T , not on V . Next, we have a formula for the entropy,

$$\left. \frac{\partial F}{\partial T} \right|_{V, N} = -S$$

$$\left. \frac{\partial F}{\partial T} \right|_{V, N} = -Nk \ln(V-b) + F_0'(T).$$

$$S = Nk \ln(V-b) - F_0'(T)$$

From this, we compute the internal energy:

$$E = F + TS = - \frac{a}{V} - NkT \ln(V-b) + F_0(T) + NkT \ln(V-b) - TF_0'(T).$$

(6)

$$\text{or } E = -\frac{a}{V} + F_0(T) - T F_0'(T).$$

Hence

$$C_V = \left. \frac{\partial E}{\partial T} \right|_{V, N} = -T F_0''(T)$$

which depends only on T .

$$(b). \quad E = -\frac{a}{V} + \int^T dT' C_V(T')$$

Let C_V be constant C_V :

$$E = -\frac{a}{V} + C_V T$$

(c) Expansion is at constant internal energy
(since isolated):

$$-\frac{a}{V_1} + C_V T_1 = -\frac{a}{V_2} + C_V T_2$$

$$\frac{a}{V_2} - \frac{a}{V_1} = C_V (T_2 - T_1) = a \frac{V_1 - V_2}{V_1 V_2}.$$

Volume increases \rightarrow T decreases.

$$V_1 = \frac{1}{3} V_2 \quad C_V (T_2 - T_1) = -\frac{2a}{V_2}.$$

(6A).

An alternative, direct, proof of (a) is as follows:

$$\begin{aligned}\left. \frac{\partial C_V}{\partial V} \right|_T &= \left(\frac{\partial}{\partial V} \left(T \frac{\partial S}{\partial T} \right)_V \right)_T \\&= T \frac{\partial^2 S}{\partial T \partial V} \\&= T \left(\frac{\partial}{\partial T} \left(\left. \frac{\partial S}{\partial V} \right|_T \right) \right)_V \\&= T \left(\frac{\partial}{\partial T} \left(\left. \frac{\partial P}{\partial T} \right|_N \right) \right)_V = T \left. \frac{\partial^2 P}{\partial T^2} \right|_V\end{aligned}$$

$$\text{But } p = \frac{NkT}{V-b} - \frac{a}{V^2}.$$

$$\Rightarrow \left. \frac{\partial^2 P}{\partial T^2} \right|_V = 0. \quad \Rightarrow C_V(T).$$

4). Dispersion relation $\epsilon = c |\vec{p}|$. in $d=3$.

(a). $N = N_0 + N_e$

$$N_0 = \frac{1}{z^{-1} - 1}$$

$g = \text{degeneracy}$
(set to $g=1$ is OK).

$$N_e = g \frac{V}{(2\pi\hbar)^3} \int d^3p \frac{1}{z^{-1} e^{\beta c p} - 1}.$$

$$= 4\pi g \frac{V}{(2\pi\hbar)^3} \int_0^\infty dp \, p^2 \frac{1}{z^{-1} e^{\beta c p} - 1}.$$

extract temperature dependence: $x \equiv \beta c p$

$$N_e = 4\pi g \frac{V}{(2\pi\hbar)^3} \left(\frac{kT}{c} \right)^3 \int_0^\infty dx \frac{x^2}{z^{-1} e^x - 1}.$$

Bose-Einstein condensation can occur if int finite for $z=1$.
The integral is related to the Riemann zeta fct:

$$\int_0^\infty dx \frac{x^2}{e^x - 1} = \sum_{n=1}^\infty \int_0^\infty dx \, x^2 e^{-nx} = 2 \zeta(3).$$

The number of excited bosons tends to 0 as $T \rightarrow 0$,
so there will be a Bose-Einstein condensation for

(b). $\frac{N}{V} = 4\pi g \left(\frac{kT_c}{2\pi\hbar c} \right)^3 2 \zeta(3).$

$$\text{or } T_c = \frac{2\pi\hbar c}{k} \left(\frac{N}{V} \frac{1}{8\pi g \zeta(3)} \right)^{1/3}.$$

(c) At temperatures below T_c , the number of excited bosons will be the maximum allowed at that temperature for $z=1$, so we have

$$N = N_0 + 4\pi g \cdot 2 \zeta(3) \left(\frac{kT}{2\pi\hbar c} \right)^3 V$$

Expressing the combination in the last term as a fct of T_c for given N, V :

$$N = 4\pi g \cdot 2 \zeta(3) \left(\frac{kT_c}{2\pi\hbar c} \right)^3 V$$

$$N = N_0 + N \frac{T^3}{T_c^3}.$$

$$\text{or } \frac{N_0}{N} = 1 - \frac{T^3}{T_c^3}.$$

(d). Modify the dimension:

$$N_e = g \frac{A}{(2\pi\hbar)^2} \int d^2p \frac{1}{z^{-1}e^{\beta c p} - 1}.$$

A = area of container.

$$N_e = g \cdot 2\pi A \left(\frac{kT}{2\pi\hbar c} \right)^2 \int_0^\infty dx \frac{x}{z^{-1}e^x - 1}.$$

This integral is still convergent, and thus bounded for $z=1$. Hence B.E. cond. occurs in $d=2$.

(9)

Note: this is in contrast with $\varepsilon = p^2/2m$:

$$N_e = g \frac{A}{(2\pi\hbar)^2} \int d^2p \frac{1}{z^{-1} e^{\beta p^2/2m} - 1}.$$

$$p = (2mkT)^{\frac{1}{2}} x^{\frac{1}{2}}$$

$$N_e = g \frac{A}{(2\pi\hbar)^2} 2mkT \cdot \pi \cdot \int_0^\infty dx \frac{1}{z^{-1} e^x - 1}.$$

The integral now diverges (logarithmically) for $z=1$.
 So, the non-relativistic BE gas does not condense in $d=2$, while the relativistic one does!

5. (a) Integrating the density of normal vibrations over all ω gives the total number of oscillators:

$$\int_0^{\omega_D} d\omega g(\omega) = \frac{3N}{\omega_D^3} \int_0^{\omega_D} \omega^2 d\omega = 3N$$

This is indeed the total number of vibrational degrees of freedom of N atoms in the solid.

- (b). The partition function for a single harmonic oscillator with frequency ω_i is given by

$$Z_i = \sum_{n=0}^{\infty} \exp(-\beta \hbar \omega_i (n + \frac{1}{2}))$$

$$= \frac{e^{-\beta \hbar \omega_i / 2}}{1 - e^{-\beta \hbar \omega_i}} = \frac{1}{2} \frac{1}{\sinh \frac{\beta \hbar \omega_i}{2}}$$

The partition function for a system of independent oscillators is the product:

$$Z = \prod_i Z_i$$

The corresponding free energy adds up

$$F = -kT \sum_i \ln Z_i$$

$$= +kT \sum_i \ln 2 \sinh \frac{\beta \hbar \omega_i}{2}$$

(11)

Given that we have here a distribution of oscillators given by the density $g(\omega) d\omega$, we get

$$F = kT \int_0^{\infty} d\omega g(\omega) \ln 2 \sinh\left(\frac{\hbar\omega}{2kT}\right).$$

The internal energy is given by

$$E = - \frac{\partial}{\partial \beta} \ln Z = \frac{\partial}{\partial \beta} \left(\int_0^{\infty} d\omega g(\omega) \ln 2 \sinh \beta \hbar \omega / 2 \right)$$

$$\frac{\partial}{\partial \beta} \ln \left(2 \sinh \frac{\beta \hbar \omega}{2} \right) = \frac{\frac{1}{2} \hbar \omega \cosh \frac{\beta \hbar \omega}{2}}{\sinh \frac{\beta \hbar \omega}{2}}$$

$$= \frac{1}{2} \hbar \omega + \frac{\hbar \omega}{e^{\beta \hbar \omega} - 1}.$$

$$E = \int_0^{\infty} d\omega g(\omega) \left[\frac{1}{2} \hbar \omega + \frac{\hbar \omega}{e^{\beta \hbar \omega} - 1} \right].$$

$$C_V = \frac{\partial E}{\partial T}$$

$$\frac{\partial}{\partial T} \frac{1}{e^{\beta \hbar \omega} - 1} = \frac{\hbar \omega / k T^2}{(e^{\beta \hbar \omega} - 1)^2} e^{\beta \hbar \omega}$$

$$C_V = \int_0^{\infty} d\omega g(\omega) (\hbar \omega)^2 / k T^2 \frac{e^{\beta \hbar \omega}}{(e^{\beta \hbar \omega} - 1)^2}.$$

For the Debye model:

$$C_v = k \frac{9N}{\omega_D^3} \int_0^{\omega_D} d\omega \omega^2 (\beta \hbar \omega)^2 \frac{e^{\beta \hbar \omega}}{(e^{\beta \hbar \omega} - 1)^2}.$$

Change variables:

$$\beta \hbar \omega = x$$

$$T_D = \frac{\hbar \omega_D}{k}. \quad \beta \hbar \omega_D = \frac{T_D}{T}$$

$$C_v = k \frac{9N}{\omega_D^3} \frac{1}{(\beta \hbar)^3} \int_0^{T_D/T} dx \frac{x^4 e^x}{(e^x - 1)^2}.$$

$$C_v = 9Nk \left(\frac{T}{T_D} \right)^3 \int_0^{T_D/T} dx \frac{x^4 e^x}{(e^x - 1)^2}.$$

(c) High T: $T_D/T \ll 1 \quad x \ll 1$

$$\int_0^{T_D/T} dx x^2 \simeq \frac{1}{3} \frac{T_D^3}{T^3}.$$

$C_v = 3Nk$, as expected from $3N$ harmonic oscillators, and equipartition.

Low T: $T_D/T \gg 1$; take $T_D/T \rightarrow \infty$
inside integral:

$$C_V = 9Nk \left(\frac{T}{T_D} \right)^3 \int_0^\infty dx \, x^4 \frac{e^x}{(e^x - 1)^2}.$$

Computing the integral: integrate by parts:

$$\begin{aligned} \int_0^\infty dx \frac{x^4 e^x}{(e^x - 1)^2} &= 4 \int_0^\infty dx \frac{x^3}{e^x - 1} \\ &= 4 \cdot \sum_{n=1}^\infty \int_0^\infty dx \, x^3 e^{-nx} = 24 \zeta(4) \end{aligned}$$

$$\zeta(4) = \frac{\pi^4}{90}.$$

$$C_V \approx \frac{12}{5} \pi^4 Nk \left(\frac{T}{T_D} \right)^3$$

