PHYSICS 210A, Winter 2009
Final Exam (100 points in total)

- Closed books and closed notes
- Please write down the necessary intermediate steps
- Write your answers in the space provided. Continue on the back if necessary.
- A formula sheet is provided in the last page of the exam.

Name:____________________________ ID: ______________________________.

Problem 1: _________________________

Problem 2: _________________________

Problem 3: _________________________

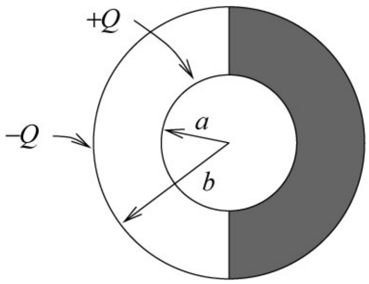
Problem 4: _________________________

Problem 5: _________________________

Problem 6: _________________________

Total: _________________________

1. Two concentric conducting spheres of inner and outer radii a and b, respectively, carry charges $\pm Q$. The empty space between the spheres is half-filled by a hemispherical shell of dielectric (of dielectric constant $\varepsilon/\varepsilon_0$) as shown in the figure.
   (a) Find the electric field everywhere between the spheres. (8 points)
   (b) Calculate the surface-charge distribution on the inner sphere. (5 points)
   (c) Calculate the bound (or polarization) charge density induced on the surface of the dielectric at $r = a$. (5 points)



2. A sphere of radius a made of linear magnetic material with permeability μ is placed in an otherwise uniform magnetic field $\vec{H}_0 = H_0\hat{z}$ in vacuum.

   (a) Find the magnetic fields, $\vec{H}$, inside and outside the sphere. (7 points)
   (b) Find the induced magnetic dipole moment and magnetization. (7 points)
   (c) Find the bound currents inside the sphere, $\vec{J}_b$, and on its surface, $\vec{K}_b$. (4 points)

3. The electric field of an electromagnetic wave in a linear medium with permeability μ has the form $\vec{E} = E_0 e^{-\alpha z} e^{-i(\omega t - kz)} \hat{y}$ where E_0 and α are real and positive quantities.
   (a) Find the $\vec{B}$ field associated with the electromagnetic wave. (6 points)
   (b) Find the time-averaged Poynting vector $\vec{S}$. (6 points)
   (c) Find the time-averaged energy density per unit time absorbed by this medium. (6 points)

4. Two infinite thin plates are located at $z = +d/2$ with potential of $+V\cos(ky)$ and at $z = -d/2$ with potential of $-V\cos(ky)$, respectively, where y is one of the coordinates parallel to the face of the plate. Find the electrostatic potential and the electric field at any point between the two plates. (15 points)

5. Consider a circular line charge of radius a in the x-y plane having a charge density

$$\lambda(\varphi) = +\lambda \qquad 0 < \varphi < \pi$$

$$\lambda(\varphi) = -\lambda \qquad \pi < \varphi < 2\pi$$

where $\varphi = \arctan(y/x)$.

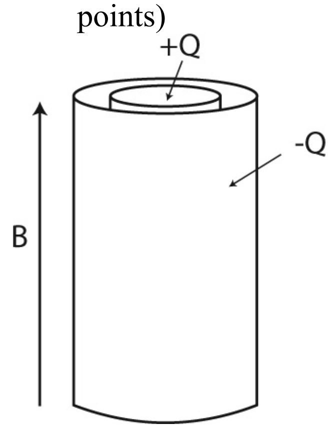
(a) Calculate the monopole moment, the dipole moment and all the components of the quadrupole moment tensor for this charge distribution. (7 points)

(b) Calculate the first two terms of the far-field potential for this charge distribution. (7 points)

6. Consider a cylindrical capacitor of length L with charge $+Q$ on the inner cylinder of radius a and $-Q$ on the outer cylindrical shell of radius b. The capacitor is filled with a lossless dielectric with dielectric constant equal to 1. The capacitor is located in a region with a uniform magnetic field $\vec{B}$, which points along the symmetry axis of the cylindrical capacitor. A flaw develops in the dielectric insulator, and a current flow develops between the two plates of the capacitor. Because of the magnetic field, this current flow results in a torque on the capacitor, which begins to rotate.

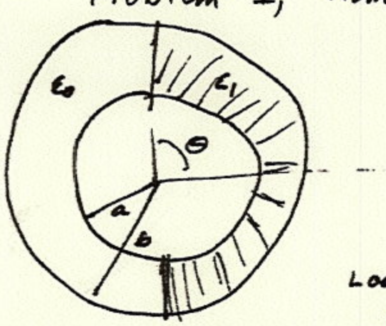
(a) After the capacitor is fully discharged (total charge on both plates is now zero), what is the magnitude and direction of the angular velocity of the capacitor? The moment of inertia of the capacitor (about the axis of symmetry) is I, and you may ignore fringing fields in the calculation. (12 points)

(b) Aside from fringing fields, what else are you ignoring in calculating the answer to (a)? (5 points)



Problem 1, from HWK [Final Solutions]

4.10

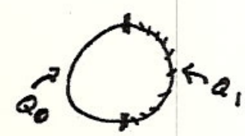


By symmetry, both the D field and the E field must be radial.

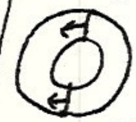
a. We can also see this by noting the boundary conditions $\Delta E_{\parallel} = 0$ $\Delta D_{\perp} = \sigma_{free}$

Look at inner conductor:
Total charge Q will divide itself into Q_0 and Q_1, evenly distributed in each half

$Q_0 + Q_1 = Q$



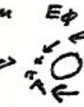
If there was flux in the $\theta = 0, \pi$ regions, by symmetry it would have to point in the same direction



not like

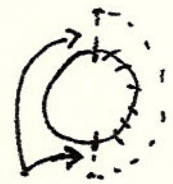
Then $E_\phi = 0$ at $\theta = \frac{\pi}{2}$.

$E_\phi = 0$ →



This would imply a divergence at that point, which can't be true since $\rho_{free} = 0$

Take a hemispherical gaussian surface



have no flux by symmetry and $\nabla \cdot E = 0$

from $\nabla \cdot D = \rho_{free}$

$$2\pi r^2 D_1 = Q_1$$

$E = \frac{1}{\varepsilon} D$

$$D_1 = \frac{Q_1}{2\pi r^2} \quad \text{and} \quad D_0 = \frac{Q_0}{2\pi r^2}$$

$$E_1 = \frac{1}{\varepsilon_1} \frac{Q_1}{2\pi r^2} \qquad E_0 = \frac{1}{\varepsilon_0} \frac{Q_0}{2\pi r^2}$$

Looking at the $\theta = 0$ or π surface and noting $E_{\parallel}$ must be continuous,

$$E_1 = E_0 \quad \Rightarrow \quad \frac{Q_1}{\varepsilon_1} = \frac{Q_0}{\varepsilon_0} \qquad Q_1 = K Q_0 \quad \text{where } K = \frac{\varepsilon_1}{\varepsilon_0}$$

combine with

$$Q_0 + Q_1 = Q$$
$$Q_0(1+K) = Q$$
$$Q_1\left(1 + \frac{1}{K}\right) = Q$$

$$\Rightarrow \boxed{Q_0 = \frac{Q}{1+K} \qquad Q_1 = \frac{KQ}{1+K}}$$

note also $\sigma_{free} = \frac{Q_{0 \text{ or } 1}}{2\pi a^2}$

so, $$\boxed{\vec{E}_0 = \frac{Q}{2\pi\varepsilon_0} \frac{1}{(1+K)r^2}\hat{r} \qquad \vec{E}_1 = \frac{Q}{2\pi\varepsilon_1} \frac{K}{(1+K)r^2}\hat{r}}$$ (which are equal)

b. The total σ $(\sigma_{free} + \sigma_{bound})$ will be the same around the inner sphere since $E = \frac{\sigma_{tot}}{\varepsilon_0}$ and E is the same in both regions

$$\sigma_{tot} = \varepsilon_0 E_0 = \boxed{\frac{Q}{2\pi a^2} \frac{1}{(1+K)}} = \sigma_{free} + \sigma_{bound}$$

from pt a, $$\sigma^0_{free} = \frac{Q}{2\pi a^2 (1+K)}$$
$$\sigma^1_{free} = \frac{Q}{2\pi a^2} \frac{K}{(1+K)}$$

c. in region 0, $\sigma_{tot} = \sigma_{free} \Rightarrow \sigma_{bound} = 0$ as expected

in region 1, $\sigma_{tot} - \sigma_{free} = \frac{Q}{2\pi a^2 (1+K)}(1-K) = \sigma_{bound}$

$$\sigma_{bound} \text{ at dielectric} = \boxed{\frac{Q}{2\pi a^2}\left(\frac{1-K}{1+K}\right)}$$

Problem 2

a Since there are no free currents $\nabla \times H = 0$ and we can use the magnetic potential formulation $-\nabla \Phi_m = H$. $\nabla \cdot B = 0$

$\Rightarrow \nabla \cdot (\mu H) = \mu \nabla \cdot H = 0$

Φ_m obeys Laplace's equation. $\nabla \cdot (-\nabla \phi) = 0 \Rightarrow \boxed{\nabla^2 \Phi_m = 0}$

$$\Phi_{in} = \sum_\ell B_\ell r^\ell P_\ell$$

manually put in the limit at ∞ $\vec{H} = H_0 \hat{z}$

$$\Phi_{out} = \sum_\ell \frac{A_\ell}{r^{\ell+1}} P_\ell - H_0 z \quad r\cos\theta = r P_1$$

$$= \left(\frac{A_1}{r^2} - H_0 r\right) P_1 + \sum_{\ell \neq 1} \frac{A_\ell}{r^{\ell+1}} P_\ell$$

Apply Boundary Condition $\Phi_{in} = \Phi_{out}$ at $r = a$

for $\ell = 1$ $B_1 a = \frac{A_1}{a^2} - H_0 a \Rightarrow \boxed{B_1 a^3 = A_1 - a^3 H_0}$ (1)

$\ell \neq 1$ $B_\ell a^\ell = \frac{A_\ell}{a^{\ell+1}}$ $\boxed{A_\ell = B_\ell a^{2\ell+1}}$ $\ell \neq 1$ (2)

Apply B.C. $B_\perp$ continuous

$B = \mu H = -\mu \nabla \phi \cdot \hat{r}$

$$\frac{\partial}{\partial r} \Phi_{in} = \sum_\ell \ell B_\ell r^{\ell-1} P_\ell$$

$$\frac{\partial}{\partial r} \Phi_{out} = \sum_{\ell \neq 1} \left[-(\ell+1) \frac{A_\ell}{r^{\ell+2}} P_\ell\right] + -2\frac{A_1}{r^3} - H_0$$

$\ell = 1$ $\mu B_1 = \mu_0 \left[\frac{-2A_1}{a^3} - H_0\right] \Rightarrow \boxed{-\frac{\mu}{\mu_0} B_1 a^3 = 2A_1 + H_0 a^3}$ (3)

$\ell \neq 1$ $\mu \ell B_\ell a^{\ell-1} = -\mu_0(\ell+1) \frac{A_\ell}{a^{\ell+2}}$ $\boxed{A_\ell = -\frac{\mu}{\mu_0} B_\ell \frac{\ell}{\ell+1} a^{2\ell+1} \quad \ell \neq 1}$ (4)

from 2, 4 $A_\ell = B_\ell = 0$ $\ell \neq 1$

from 1, 3 $A_1 = -a^3 H_0 \dfrac{1 - \mu/\mu_0}{2 + \mu/\mu_0}$ $B_1 = -\dfrac{3H_0}{2 + \mu/\mu_0}$

So $$\Phi_{in} = -\frac{3H_0}{2 + \mu/\mu_0} r\cos\theta$$

$$\Phi_{out} = -H_0 r\cos\theta \left[1 + \frac{a^3}{r^3} \frac{1 - \mu/\mu_0}{2 + \mu/\mu_0}\right]$$

$$\vec{H}_{in} = \frac{1}{\mu}\vec{B}_{in} = -\nabla\Phi_{in} = \frac{3H_0}{2+\mu/\mu_0}\left[\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\right] = \boxed{\frac{3H_0}{2+\mu/\mu_0}\hat{z}}$$

$$\vec{H}_{out} = \frac{1}{\mu_0}\vec{B}_{out} = \boxed{H_0\hat{z} - H_0 \frac{a^3}{r^3}\frac{1-\mu/\mu_0}{2+\mu/\mu_0}\underbrace{\left[2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right]}_{\text{dipole field}}}$$

b. $\Phi_{dipole} = \frac{\vec{m}\cdot\hat{r}}{4\pi r^2}$

compare to $\Phi_{out} = -H_0 r\cos\theta \frac{a^3}{r^3}\frac{1-\mu/\mu_0}{2+\mu/\mu_0}$ + uniform field

$\vec{m}$ is in $\hat{z}$ with magnitude

$$\boxed{|\vec{m}| = -4\pi H_0 a^3 \frac{(1-\mu/\mu_0)}{2+\mu/\mu_0}}$$

Magnetization, $\vec{M} = \chi\vec{H}$

remember, $\mu = \mu_0(1+\chi)$

$$= \left(\frac{\mu}{\mu_0}-1\right)\vec{H}$$

$$= \boxed{-3H_0 \frac{1-\mu/\mu_0}{2+\mu/\mu_0}\hat{z}}$$

compare $\vec{m}$ to $\vec{M}$, we see that $\vec{m} = \frac{4}{3}\pi a^3\vec{M}$, as it should be.

c. $\vec{J}_b = \nabla\times\vec{M}$

$\vec{M}$ is constant so $\boxed{\vec{J}_b = 0}$

$\vec{K}_b = \vec{M}\times\hat{n} = M\hat{z}\times\hat{r} = M(\cos\theta\hat{r} - \sin\theta\hat{\theta})\times\hat{r}$

$$= M\sin\theta\,\hat{\phi}$$

$$= \boxed{-3H_0 \frac{1-\mu/\mu_0}{2+\mu/\mu_0}\sin\theta\,\hat{\phi}}$$

Problem 3

a. $\vec{E} = E_0 e^{-az} e^{i(kz-\omega t)} \hat{y}$ where E_0 and a are real + positive

take a to be the imaginary part of a complex K vector

$\tilde{K} = K + ia$ then $e^{i\tilde{K}z} = e^{ikz} e^{-az}$ as above

$\vec{E} = E_0 e^{i(\tilde{K}z-\omega t)} \hat{y}$

Since this is traveling in the z direction, $E\times B$ must point in $\hat{z}$

so, B must be in $-\hat{x}$

$\vec{B} = B_0 e^{i(\tilde{K}z-\omega t)} (-\hat{x})$ 3pts

we still need to solve for B_0

From Maxwell, $\nabla\times E = -\frac{\partial B}{\partial t}$

$\nabla\times E = -\frac{\partial}{\partial z} E_y \hat{x} + \underbrace{\frac{\partial}{\partial x} E_y \hat{z}}_{=0}$ $\qquad \frac{\partial B_x}{\partial t} = -i\omega B_x$

$= i\tilde{K} E_y$

$\Rightarrow \quad i\tilde{K}E_y = i\omega B_x$

$B_x = \frac{\tilde{K}}{\omega} E_y$ so $\boxed{B_0 = \frac{\tilde{K}}{\omega} E_0}$ 3 pts

$$\boxed{\vec{B} = \frac{\tilde{K}}{\omega} E_0 e^{i(\tilde{K}z-\omega t)} (-\hat{x})}$$

b. for complex fields, $\langle \vec{S} \rangle = \mathrm{Re}\left[\frac{1}{2} \vec{E}\times\vec{H}^*\right]$ (time average)

so the easy way to do this is to plug in.

$$\langle\vec{S}\rangle = \frac{1}{2}\mathrm{Re}\left[E_0 e^{-az} e^{i(kz-\omega t)} \frac{\tilde{K}^*}{\mu\omega} E_0 e^{-az} e^{-i(kz-\omega t)}\right] = \frac{1}{2\mu}E_0^2 e^{-2az}\mathrm{Re}\left[\frac{\tilde{K}}{\omega}\right]$$

$$= \boxed{\frac{1}{2\mu}\frac{K}{\omega}E_0^2 e^{-2az}\hat{z}}$$

if you did not know about $\langle\vec{S}\rangle = \frac{1}{2}\mathrm{Re}\,\vec{E}\times\vec{H}^*$ (complex) and you used the instantaneous Poynting vector $\vec{S} = \vec{E}\times\vec{H}$ (Real) the algebra is more intensive, here it is...

b. longer way

$$\vec{E} = \mathrm{Re}\, E_0 e^{-az} e^{i(kz-\omega t)} \hat{y} = E_0 e^{-az} \cos(kz-\omega t)\hat{y}$$

$$\vec{B} = \mathrm{Re}\left[-\frac{\tilde{k}}{\omega} E_0 e^{-az} e^{i(kz-\omega t)}\right.$$

$$= \mathrm{Re}\frac{-E_0}{\omega}(k+ia) e^{-az}\hat{x}\left(\cos(kz-\omega t) + i\sin(kz-\omega t)\right)$$

$$= -\frac{E_0}{\omega} e^{-az}\hat{x}\left[k\cos(kz-\omega t) - a\sin(kz-\omega t)\right]$$

$$\vec{S} = \frac{1}{\mu}\vec{E}\times\vec{B} = +\frac{E_0^2}{\mu\omega} e^{-2az}\hat{z}\left[k\cos^2(kz-\omega t) - a\cos(kz-\omega t)\sin(kz-\omega t)\right]$$

Now, time average the two terms

$$\frac{1}{2\pi}\int_0^{2\pi}\cos^2 x\, dx = \frac{1}{2} \qquad \frac{1}{2\pi}\int_0^{2\pi}\sin x\cos x = 0$$

so,

$$\boxed{\langle\vec{S}\rangle = \frac{E_0^2}{2\mu}\frac{k}{\omega} e^{-2az}\hat{z}}$$

c.

$$\nabla\cdot S + \frac{\partial}{\partial t}\left(u_{mech} + u_{field}\right) = 0$$

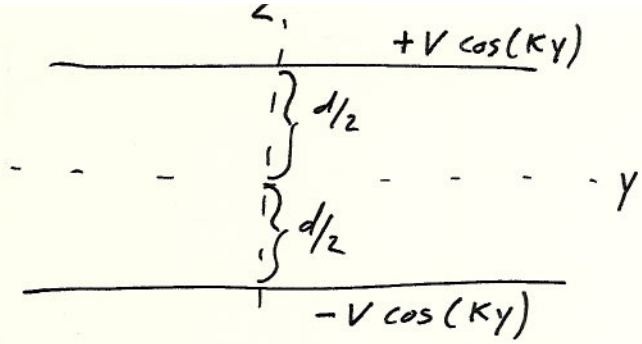
energy density absorbed

=0 since the fields do not change on time average

$$\frac{\partial}{\partial t}(u_{mech}) = -\nabla\cdot S = -(-2a)\frac{E_0^2}{2\mu}\frac{k}{\omega} e^{-2az}$$

$$\boxed{\frac{\partial u_{mech}}{dt} = E_0^2 \frac{ak}{\mu\omega} e^{-2az}}$$

Problem 4



Let $\Phi = R(z)S(y)$

$\nabla^2\Phi = 0 \quad \Rightarrow \quad \frac{1}{R}\frac{\partial^2 R}{\partial z^2} + \frac{1}{S}\frac{\partial^2 S}{\partial y^2} = 0$ so each term must be constant.

We know the solution oscillates in y so pick negative constant for S

$$\frac{1}{S}\frac{\partial^2 S}{\partial y^2} = -m^2 \qquad \frac{1}{R}\frac{\partial^2 R}{\partial z^2} = m^2$$

solutions are $\phi(z,y) = \sum_m \left(A_m \cos(my) + B_n \sin(my)\right)\left(C_m \cosh mz + D_n \sinh mz\right)$

we know ϕ is odd in z and even in y, so

$$\phi(z,y) = A_m \cos(my)\sinh(mz)$$

at $z = \pm d/2$

$$\phi\left(\pm\frac{d}{2}, y\right) = A_m \cos(my)\sinh\left(\pm m\frac{d}{2}\right) = \pm V\cos(Ky)$$

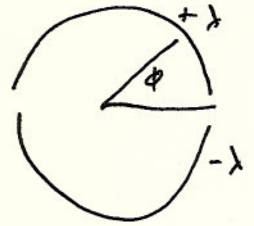
we see $A_m \sinh\left(\pm m\frac{d}{2}\right) = \pm V$ and $K = m$

$$A_m = \frac{V}{\sinh\left(K\frac{d}{2}\right)}$$

$$\boxed{\phi(z,y) = \frac{V}{\sinh\left(\frac{Kd}{2}\right)}\cos(Ky)\sinh(Kz)}$$

$$\boxed{\vec{E} = -\nabla\phi = -\frac{VK}{\sinh\left(\frac{Kd}{2}\right)}\left[\cosh(Kz)\cos(Ky)\,\hat{z} - \sinh(Kz)\sin(Ky)\,\hat{y}\right]}$$

Problem 5



a. Monopole = 0 because total charge is zero

dipole: by symmetry $P_x = P_z = 0$

$$\vec{P} = \int \vec{x}\,'\rho(x')\,d^3x$$

$$P_y = \pm\lambda \int y \; a\,d\phi$$

$$= \lambda a\left[\int_0^{\pi} y\,d\phi - \int_{\pi}^{2\pi} y\,d\phi\right] = \lambda a^2\left[\int_0^{\pi}\sin\phi - \int_{\pi}^{2\pi}\sin\phi\right]$$

$$= 2\lambda a^2\left[\int_0^{\pi}\sin\phi\right] = 4\lambda a^2 \qquad \boxed{P_y = 4\lambda a^2}$$

4 pts

Quadrupole: $Q_{ij} = \int\left[3x_i x_j - r'^2\delta_{ij}\right]\rho(\vec{r}\,')\,d^3x$

write the charge density in terms of δ fns

3 pts for rest of pt a

$$\rho(\vec{r}) = \pm N\cdot\lambda\,\delta(r-a)\,\delta(\cos\theta) \qquad \begin{matrix} + & \text{for } y>0 \\ - & \text{for } y<0\end{matrix}$$

here N is a normalization I always have to solve for.

If we integrate ρ in the top half of the plane, we should get $\pi a\lambda$

so, $$q_{Top} = \pi a\lambda = \int_{\text{Top half}} \rho(\vec{x})\,r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= \lambda N\int\delta(r-a)\,\delta(\cos\theta)\,r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= \lambda N a^2\pi \qquad \text{so} \qquad N = \frac{1}{a}$$

$$\rho(\vec{x}) = \pm\frac{\lambda}{a}\,\delta(r-a)\,\delta(\cos\theta)$$

$$Q_{ij} = \int\left[3x_i x_j - r'^2\delta_{ij}\right]\pm\frac{\lambda}{a}\,\delta(r-a)\,\delta(\cos\theta)\,r^2\sin\theta\,dr\,d\theta\,d\phi$$

$Q_{xz} = Q_{zx} = Q_{yz} = Q_{zy} = 0$ because $\delta_{ij} = 0$ and $\delta(\cos\theta)$ forces $z=0$

$$Q_{xy} = Q_{yx} = \int\left[3r^2\sin^2\theta\cos\phi\sin\phi\right]\pm\frac{\lambda}{a}\,\delta(r-a)\,\delta(\cos\theta)\sin\theta\,r^2\,dr\,d\theta\,d\phi$$

$$= \frac{\lambda}{a}a^4\left[\int_0^{\pi}3\cos\phi\sin\phi\,d\phi - \int_{\pi}^{2\pi}3\cos\phi\sin\phi\,d\phi\right]$$ each integral is zero

$$= 0$$

$$Q_{xx} = \frac{\lambda}{a} a^4 \left[\int_0^{\pi} (3\cos^2\phi - 1)\, d\phi - \int_{\pi}^{2\pi} (3\cos^2\phi - 1)\, d\phi \right]$$

let $u = \phi - \pi$
$du = d\phi$
$\cos\phi = \cos(u+\pi) = -\cos u$
$\cos^2\phi = \cos^2 u$

$$\int_0^{\pi} (3\cos^2\phi - 1)\, d\phi - \int_0^{\pi} (3\cos^2 u - 1)\, d\phi$$

$$= 0$$

$$Q_{yy} = \frac{\lambda}{a} a^4 \left[\int_0^{\pi} (3\sin^2\phi - 1)\, d\phi - \int_{\pi}^{2\pi} (3\sin^2\phi - 1)\, d\phi \right] = 0$$ by same argument

$$Q_{zz} = \frac{-\lambda}{a} a^4 \left[\int_0^{\pi} 1\, d\phi - \int_{\pi}^{2\pi} d\phi \right] = 0$$

$$\boxed{Q_{ij} = 0 \quad \text{for all } i,j}$$

b. the dipole term is the first term, which we already have. Since there is no Quadrupole term, we need to find the octopole term.

Now, We could solve $q_{lm} = \int Y^*_{lm}(\theta', \phi') r'^{l} \rho(x')\, d^3x$ for all m but that is long. I'm going to take another approach.

$$\Phi = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho}{|x-x'|}\, d^3x \qquad |x-x'| = \left[(x-x')^2 + (y-y')^2 + (z-z')^2 \right]^{1/2} \text{ etc...}$$

from law of cosine $|x-x'|^2 = r^2 + a^2 - 2ra\cos\gamma$ where γ is the angle between the vectors

$$\cos\gamma = \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos(\phi_1 - \phi_2)$$

for us, let $\theta_1 = \theta$, $\phi_1 = \phi$; $\theta_2 = \theta' = \frac{\pi}{2}$, $\phi_2 = \phi'$ $\Big\}$ $\cos\gamma = \sin\theta\cos(\phi - \phi')$

$$\Phi = \frac{\lambda}{4\pi\varepsilon_0} \left[\int_0^{\pi} \frac{a\, d\phi}{|x-x'|} - \int_{\pi}^{2\pi} \frac{a\, d\phi}{|x-x'|} \right]$$ let us expand $\frac{1}{|x-x'|}$

$$\frac{1}{|x-x'|} = \frac{1}{r}\Big[1 + \underbrace{\frac{a^2}{r^2} - \frac{2a}{r}\cos\gamma}_{\text{small parameter } \varepsilon}\Big]^{-1/2} \qquad (1+\varepsilon)^{-1/2} = 1 - \frac{\varepsilon}{2} + \frac{3}{8}\varepsilon^2 - \frac{15}{48}\varepsilon^3$$

Since we want the octupole term we expand $\frac{1}{|x-x'|}$ to $\frac{1}{r^4}$, we want all terms in the expansion to $\frac{1}{r^3}$, so up to ε^3

$$\varepsilon = \frac{a^2}{r^2} - \frac{2a}{r}\cos\gamma$$

$$\varepsilon^2 = \frac{a^4}{r^4} - 4\frac{a^3}{r^3} + 4\frac{a^2}{r^2}\cos^2\gamma$$

~~$\varepsilon^3 = 4\frac{a^4}{r^4}\cos^2\gamma + 8\frac{a^4}{r^4}\cos\gamma - 8$~~

$$\varepsilon^3 = 4\frac{a^4}{r^4}(\cos^2\gamma + 2\cos\gamma) - \frac{8a^3}{r^3}\cos^3\gamma$$

only need up to $\frac{1}{r^3}$ terms

I wrote up to $\frac{1}{r^4}$ though

since we only care about octupole you only need this term

$$\left[1+\frac{a^2}{r^2}-\frac{2a}{r}\cos\gamma\right]^{-1/2} \approx 1+\frac{a}{r}\cos\gamma+\frac{a^2}{r^2}\left(-\frac{1}{2}+\frac{3}{2}\cos^2\gamma\right)+\frac{a^3}{r^3}\left(-\frac{3}{2}+\frac{5}{2}\cos^3\gamma\right)+\dots$$

but I will do all terms just for thoroughness

$$\Phi = \frac{\lambda a}{4\pi\varepsilon_0}\frac{1}{r}\int\left[1+\frac{a^2}{r^2}-\frac{2a}{r}\cos\gamma\right]^{-1/2}$$

$$\Phi_0 = \frac{\lambda a}{4\pi\varepsilon_0}\frac{1}{r}\int \pm 1\, d\phi' = \frac{\lambda a}{4\pi\varepsilon_0 r}\left[\int_0^{\pi} d\phi - \int_{\pi}^{2\pi} d\phi'\right] = 0 \qquad \boxed{\text{monopole} = 0}$$

$$\Phi_1 = \frac{\lambda a}{4\pi\varepsilon_0 r}\left(+\frac{a}{r}\right)\left[\int_0^{\pi}\cos\gamma\, d\phi' - \int_{\pi}^{2\pi}\cos\gamma\, d\phi'\right]$$

let $u = \phi' - \pi$

$du = d\phi'$

$\cos(\phi-\phi') \to -\cos(\phi - u)$

remember $\cos\gamma = \sin\theta\cos(\phi-\phi')$

$\cos\gamma \to -\cos\gamma$ for $\phi' \to \phi' - \pi$

$$\left[\int_0^{\pi}\cos\gamma\, d\phi - \int_{\pi}^{2\pi}\cos\gamma\, d\phi\right] = 2\int_0^{\pi}\cos\gamma = 2\sin\theta\underbrace{\int_0^{\pi}\cos(\phi-\phi')\,d\phi'}_{+2\sin\phi} = +4\sin\theta\sin\phi$$

$$= \frac{\lambda a^2}{4\pi\varepsilon_0 r^2}4\sin\theta\sin\phi = \boxed{\frac{4\lambda a^2}{4\pi\varepsilon_0 r^2}\sin\theta\sin\phi}$$

← 2 pts for this

dipole $\frac{\vec{p}\cdot\hat{x}}{4\pi\varepsilon_0 r^2}$ $\quad P_y = 4\lambda a^2$ same as before

$$\Phi_2 = \frac{\lambda a}{4\pi\varepsilon_0 r}\frac{a^2}{2r^2}\left[\int_0^{\pi}(-1+3\cos^2\gamma)\,d\phi' - \int_{\pi}^{2\pi}(-1+3\cos^2\gamma)\,d\phi'\right]$$

doing the same substitution above, the integrals are identical and cancel

$$= 0 \qquad \boxed{\text{Quadrupole} = 0}$$

This is the only term you needed to do:

$$\Phi_3 = \frac{\lambda a}{4\pi\varepsilon_0 r}\left(\frac{a^3}{2r^3}\right)\left[\underbrace{\int_0^{\pi}(-3+5\cos^3\gamma) - \int_{\pi}^{2\pi}(-3+5\cos^3\gamma)}\right]$$

same substitution
the -3 terms cancel
the $\cos^3\gamma$ terms add

$$2\int_0^{\pi}5\cos^3\gamma\, d\phi' = 10\sin^3\theta\int_0^{\pi}\cos^3(\phi-\phi')\,d\phi'$$

$$\int_0^{\pi}\cos^3(\phi-\phi')\,d\phi = -\int\cos(\sin^2-1) = \int_0^{\pi}\cos(\phi-\phi')\,d\phi - \int\cos(\phi-\phi')\sin^2(\phi-\phi')$$

$u = \phi - \phi'$ $\quad u = \sin(\phi-\phi')$

$$= \int_{\phi}^{\phi-\pi}-\cos u\, du + \int_{\sin\phi}^{\sin\phi-\pi}u^2\, du = 2\sin\phi - \frac{2}{3}\sin^3\phi$$

$$\boxed{= \frac{\lambda a^4}{4\pi\varepsilon_0 r^4}10\sin^3\theta\left(\sin\phi - \frac{1}{3}\sin^3\phi\right)}$$

← 5 pts for this octupole ... finally

5b. easier way

So I just realized the expansion I just did is...

$$\frac{1}{|x-x'|} \approx \frac{1}{r}\left[\underbrace{1}_{P_0} + \frac{a}{r}\underbrace{\cos\gamma}_{P_1} + \frac{a^2}{r^2}\underbrace{\left(-\frac{1}{2}+\frac{3}{2}\cos^2\gamma\right)}_{P_2} + \frac{a^3}{r^3}\underbrace{\left(-\frac{3}{2}+\frac{5}{2}\cos^3\gamma\right)}_{P_3} + \cdots\right]$$

$$= \sum_l \frac{r_<^l}{r_>^{l+1}} P_l(\cos\gamma)$$

ugh, all that algebra for no reason, I'm dumb.

All you needed to do then is take $l=3$ of

$$\phi = \frac{1}{4\pi\varepsilon_0}\int \frac{\pm\lambda}{|x-x'|} = \frac{\lambda}{4\pi\varepsilon_0}\sum_l \frac{a^l}{r^{l+1}}\left[\int_0^\pi P_l\,d\phi' - \int_\pi^{2\pi} P_l\,d\phi'\right]$$

$$\left[\int_0^\pi P_l(\cos\gamma) - \int_0^\pi P_l(-\cos\gamma)\right]$$

remember $P_l(x)$ is even for ~~odd~~ even l
odd for ~~even~~ odd l so even terms die

$$\phi = \frac{\lambda}{4\pi\varepsilon_0}\,\frac{2}{r}\sum_{l\,\text{odd}}\left(\frac{a}{r}\right)^l \int_0^\pi P_l(\cos\gamma)\,d\phi'$$

so plug in $l=3$, integrate and you get your octopole term
quick + dirty

Problem 6

a.

$$\vec{L} = \int \ell_{field} = \frac{\varepsilon_0\mu_0}{\mu} \int \vec{r} \times \left[\vec{E} \times \vec{B} \right] d^3x$$

$$= \frac{\varepsilon_0\mu_0 BQ}{\mu \cdot 2\pi\varepsilon L} \int \frac{1}{\rho} \vec{r} \times \underbrace{[\hat{\rho} \times \hat{z}]}_{-\hat{\phi}}$$

$$= \frac{-\varepsilon_0\mu_0}{\mu\varepsilon} \frac{BQ}{2\pi L} \int \frac{dz}{\rho} \rho d\rho \, d\phi \left[\rho\cos^2\phi + \rho\sin^2\phi \right] \hat{z}$$

$$\int dz d\rho d\phi \, \rho \hat{z}$$

$$= \frac{-\varepsilon_0\mu_0}{\varepsilon\mu} \frac{BQ}{2\pi L} L \cdot 2\pi \left. \frac{\rho^2}{2} \right|_a^b$$

$$= \frac{-\varepsilon_0\mu_0}{\varepsilon\mu} \frac{BQ}{2} (b^2 - a^2) \hat{z}$$

$$\vec{E} = \frac{Q}{2\pi\varepsilon L} \frac{1}{\rho} \hat{\rho}$$

$$\vec{B} = B_0 \hat{z}$$

$$\vec{r} \times \hat{\phi} = (x\hat{x} + y\hat{y} + z\hat{z}) \times (-\sin\phi\hat{x} + \cos\phi\hat{y})$$

$$= (\underset{\nearrow \, \rho\cos\phi}{x\cos\phi} + \underset{\nearrow \, \rho\sin\phi}{y\sin\phi})\hat{z} \underbrace{- z\sin\phi\hat{y} - z\cos\phi\hat{x}}_{\text{integrate to zero b.c. of } \phi}$$

this angular momentum all gets transferred to the cylinder

$$I\omega = L \qquad \boxed{\vec{\omega} = -\frac{BQ}{2I} \frac{\varepsilon_0\mu_0}{\varepsilon\mu} (b^2 - a^2) \hat{z}}$$

b. We ignore angular momentum lost to radiation.

PHYSICS 210A, Winter 2009
Midterm Exam (100 points in total)

- Closed books and closed notes
- Please write down the necessary intermediate steps
- Write your answers in the space provided. Continue on the back if necessary.
- A formula sheet is provided at the end of the exam.

Name:____________________________ ID: ______________________________.

Problem 1: _________________________

Problem 2: _________________________

Problem 3: _________________________

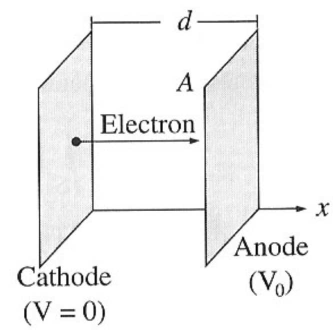
Problem 4: _________________________

Total: _________________________

1. (a) A charged sphere of radius a has a uniform charge density within its volume with a total charge Q. Calculate the electric fields inside and outside the sphere. (10 points)
   (b) Assume the charge density distribution in (a) is spherically symmetric and varies radially as r^n ($n > -3$). Calculate the electric fields inside and outside the sphere. (10 points)

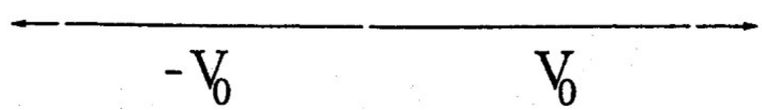
(b) In a vacuum diode, electrons are boiled off a hot cathode, at potential zero, and accelerated across a gap to the anode, which is held at positive potential V_0 (see the figure below). The cloud of moving electrons within the gap (called the space charge) quickly builds up to the point where it reduces the field at the surface of the cathode to zero. From then a steady current I flows between the plates. Suppose the plates are large relative to the separation ($A >> d^2$), so that edge effects can be neglected. Then V, ρ, and v (the speed of the electrons) are all functions of x alone.

a. Assuming the electrons start from rest at the cathode, what is their speed at point x, where the potential is $V(x)$? (5 points)
b. In the steady state, I is independent of x. What is the relation between ρ and v? (5 points)
c. Use the results in (a) and (b) to obtain a differential equation for V, by eliminating ρ and v. (5 points)
d. Solve this equation for V as a function of x, V_0 and d. (Hint: you may use the identity, $\frac{d\Phi}{dx}\frac{d^2\Phi}{dx^2} = \frac{1}{2}\frac{d}{dx}\left(\frac{d\Phi}{dx}\right)^2$.) (10 points)
e. Show that $I = kV_0^{3/2}$ and find the constant k. This equation is called the *Child-Langmuir law*. It holds for other geometries as well, whenever space-charge limits the current. Notice that the space-charge limited diode is nonlinear and doesn't obey the *Ohm's law*.) (5 points)



(c) A sphere of radius a has a surface charge density $\sigma = \sigma_0 \cos(2\theta)$. Find the potential at all points in space exterior and interior to the sphere. (25 points)

(d) Consider the infinite two-dimensional conducting plane depicted in the figure below. The right half is maintained at electrostatic potential V_0 while the left half is maintained at potential $-V_0$. Calculate the potential above the plane. (25 points)



Formula Sheet for Midterm

I decided to provide you most the equations in Chapters 1 - 3 (much more than you need in the midterm). I want you to understand the physics instead of memorizing the equations.

$$\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{r^2}\hat{r} \qquad \vec{E} = -\vec{\nabla}\Phi \qquad \vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \qquad \nabla^2\Phi = -\frac{\rho}{\varepsilon_0}$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\int\frac{\hat{r}}{r^2}\rho(\vec{x}')dV' \qquad \Phi = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\vec{x}')}{r}dV' \qquad W = \frac{1}{2}\int\rho\Phi dV = \frac{\varepsilon_0}{2}\int|E|^2\,dV$$

$$C = \frac{Q}{V} \qquad W = \frac{1}{2}CV^2 \qquad \Phi_1 = \Phi_2 \qquad \frac{\partial\Phi_2}{\partial n} - \frac{\partial\Phi_1}{\partial n} = -\frac{\sigma}{\varepsilon_0}$$

$$\nabla^2 G(\vec{x},\vec{x}') = -4\pi\delta(\vec{x}-\vec{x}') \qquad G(\vec{x},\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|} + F(\vec{x},\vec{x}')$$

$$\Phi = \frac{1}{4\pi\varepsilon_0}\int_V G_D(\vec{x},\vec{x}')\rho(\vec{x}')dV' - \frac{1}{4\pi}\oint_S \Phi\frac{\partial G_D(\vec{x},\vec{x}')}{\partial n'}da'$$

$$\Phi = \frac{1}{4\pi\varepsilon_0}\int_V G_N(\vec{x},\vec{x}')\rho(\vec{x}')dV' + \frac{1}{4\pi}\oint_S G_N(\vec{x},\vec{x}')\frac{\partial\Phi}{\partial n'}da'$$

$$\frac{1}{|\vec{x}-\vec{x}'|} = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_l^{m*}(\theta',\varphi')Y_l^m(\theta,\varphi)$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}a_n\cos n\frac{\pi x}{L} + \sum_{n=1}^{\infty}b_n\sin n\frac{\pi x}{L} \qquad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos n\frac{\pi x}{L}\,dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin n\frac{\pi x}{L}\,dx$$

$$g(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx \qquad f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(k)e^{ikx}dk$$

(i) Rectangular coordinates

$$\nabla^2\Phi(x,y,z) = 0 \qquad \Phi(x,y,z) \sim e^{\pm ik_x x}e^{\pm ik_y y}e^{\pm k_z z}, \quad k_z^2 = k_x^2 + k_y^2$$

(ii) 2D Polar Coordinates

$$\Phi(r,\varphi) = a_0 + b_0\ln r + \sum_{n=1}^{\infty}\left[a_n r^n\sin(n\varphi+\alpha_n) + b_n r^{-n}\sin(n\varphi+\beta_n)\right]$$

(iii) Spherical Coordinates with azimuthal symmetry (m = 0)

$$\Phi(r,\theta) = \sum_{l=0}^{\infty}\left[A_l r^l + \frac{B_l}{r^{l+1}}\right]P_l(\cos\theta) \qquad \int_{-1}^{1}P_l(x)P_{l'}(x)dx = \frac{2}{2l+1}\delta_{ll'}$$

$$P_0(x) = 1 \qquad P_1(x) = x \qquad P_2(x) = \frac{1}{2}(3x^2-1) \qquad P_3(x) = \frac{1}{2}(5x^3-3x)$$

(iv) Spherical coordinates (m ≠ 0)

$$\Phi(r,\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{m}\left[A_l^m r^l + \frac{B_l^m}{r^{l+1}}\right]Y_l^m(\theta,\varphi) \qquad \int Y_l^m(\theta,\varphi)Y_{l'}^{m'*}(\theta,\varphi)d\Omega = \delta_{ll'}\delta_{mm'}$$

(v) Cylindrical Coordinates

$$\Phi(r,\varphi,z) = \sum_{\nu=0}^{\infty}\left[A_\nu J_\nu(kr) + B_\nu N_\nu(kr)\right]e^{\pm i\nu\varphi}\, e^{\pm kz} \qquad \int_0^a rJ_\nu\left(\frac{x_{\nu n'}r}{a}\right)J_\nu\left(\frac{x_{\nu n}r}{a}\right)dr = \frac{a^2}{2}[J_{\nu+1}(x_{\nu n})]^2\delta_{nn'}$$

1a. $\rho = \frac{Q}{\frac{4}{3}\pi R^3}$

$$E_{in} \cdot 4\pi r^2 = \frac{\rho \cdot \frac{4}{3}\pi r^3}{\varepsilon_0}$$

$$E_{in} = \frac{\cancel{4}}{3}\cancel{\pi}\,\frac{r}{\varepsilon_0}\,\rho$$

$$= \frac{r}{3\varepsilon_0}\,\frac{Q}{\frac{4}{3}\pi R^3}$$

$$= \boxed{\frac{Q}{4\pi\varepsilon_0}\,\frac{r}{R^3}}$$

$$\boxed{E_{out} = \frac{Q}{4\pi\varepsilon_0}\,\frac{1}{r^2}}$$

1b. $\rho = \rho_0 r^n$

$$Q = \int \rho\, d^3x = 4\pi \int \rho_0 r^n r^2 dr = 4\pi\rho_0 \int_0^R r^{n+2} dr$$

$$= 4\pi\rho_0 \frac{R^{n+3}}{n+3}$$

$$\boxed{Q = \frac{4\pi\rho_0}{n+3} R^{n+3}}$$

$$\rho_0 = \frac{(n+3)Q}{4\pi R^{n+3}}$$

$$4\pi r^2 E_{in} = \frac{4\pi}{\varepsilon_0}\int_0^r \rho r^2 dr = \frac{4\pi}{\varepsilon_0}\rho_0 \frac{r^{n+3}}{(n+3)} = \frac{Q}{\varepsilon_0}\frac{r^{n+3}}{R^{n+3}}$$

$$\boxed{E_{in} = \frac{Q}{4\pi\varepsilon_0}\,\frac{r^{n+1}}{R^{n+3}} \qquad E_{out} = \frac{Q}{4\pi\varepsilon_0}\,\frac{1}{r^2}}$$

2.

a. $\frac{1}{2}Mv^2 = q\,V(x)$

$$v = \sqrt{\frac{2qV(x)}{M}} \qquad V = \frac{1}{2}\frac{M}{q}v^2$$

b. Number density; cross sectional Area; thickness

$$I = \frac{Ne\cdot q}{t} = \frac{\rho_N \cdot A \cdot \Delta t\, q}{t} \qquad \frac{\Delta t}{t} = v$$

thickness; time

$$= \rho_N q A v$$

so, $\boxed{\rho_N = \frac{I}{qA}\frac{1}{v(x)}}$ or $\rho = \frac{I}{A}\frac{1}{v(x)}$

c. $\nabla^2 V = -\frac{\rho}{\varepsilon_0}$

$$\frac{\partial^2 V}{\partial x^2} = \frac{-q}{\varepsilon_0}\rho_N$$

$$\frac{\partial^2}{\partial x^2}V = \frac{-q}{\varepsilon_0}\frac{I}{qA}\frac{1}{v(x)}$$

$$= -\frac{I}{A\varepsilon_0}\sqrt{\frac{M}{2qV(x)}}$$

$$= \underbrace{-\frac{I}{A\varepsilon_0}\sqrt{\frac{M}{2q}}}_{K}\, V^{-1/2}$$

$$\boxed{\frac{\partial^2 V}{\partial x^2} = -K\,V^{-1/2}}$$

d. solve with boundary conditions $V(0)=0$, $V(d)=V_0$

Guess $V(x) = C\cdot x^{\nu}$

$$\frac{\partial V}{\partial x} = C\nu x^{\nu-1}$$

$$\frac{\partial^2 V}{\partial x^2} = C\nu(\nu-1)x^{\nu-2}$$

$$C\nu(\nu-1)x^{\nu-2} = -K\left[Cx^{\nu}\right]^{-1/2} = -KC^{-1/2}x^{-\nu/2}$$

Thus: $x^{\nu-2} = x^{-\nu/2}$

and $c\,\nu(\nu-1) = -K c^{-1/2}$

$\nu - 2 = -\frac{\nu}{2}$

$2\nu - 4 = -\nu$

$3\nu = 4$

$\nu = \cancel{4/4}\ 4/3$

$c^{3/2}\ \frac{4}{3}\left(\frac{1}{3}\right) = -K$

$c^{3/2} = -K\,\frac{9}{4}$ $\qquad c = \left(-K\,\frac{9}{4}\right)^{2/3}$

$$V(x) = \left(\frac{81}{16}K^2\right)^{1/3} x^{4/3}$$

$$= \left(\frac{81}{16}\,\frac{I^2}{A^2\varepsilon_0^{\,2}}\,\frac{M}{2q}\right)^{1/3} x^{4/3}$$

$$V(d) = V_0 = \left(\frac{81}{16}\,\frac{I^2}{A^2\varepsilon_0^2}\,\frac{M}{2q}\right)^{1/3} d^{4/3}$$

$$V(x) = \frac{V_0}{d^{4/3}}\,x^{4/3} = \boxed{V_0\left(\frac{x}{d}\right)^{4/3}}$$

e.

solve for I: $\left(\frac{V_0}{d^{4/3}}\right)^3 = \frac{81}{16}\,\frac{I^2}{A^2\varepsilon_0^2}\,\frac{M}{2q}$

$$I^2 = \frac{32}{81}A^2\varepsilon_0^{\,2}\,\frac{q}{M}\left(\frac{V_0^3}{d^4}\right)$$

$$I = \sqrt{\frac{32}{81}A^2\varepsilon_0^2\,\frac{q}{M}\,\frac{1}{d^4}}\ V_0^{3/2}$$

$$\boxed{I = \frac{4}{9}\,\frac{A\varepsilon}{d^2}\sqrt{2\frac{q}{M}}\ V_0^{3/2}}$$

Method 1 : Sep of Var

3.

$\sigma = \sigma_0 \cos(2\theta)$

$$\phi_{out} = \sum_l \frac{B_l}{r^{l+1}} P_l$$

$$\phi_{in} = \sum_l A_l r^l P_l$$

$$\phi_{in} = \phi_{out} \Rightarrow \sum_l \frac{B_l}{a^{l+1}} P_l = \sum_l A_l a^l P_l$$

$$\boxed{B_l = A_l a^{2l+1}} \qquad (1)$$

$$\frac{\partial \phi_{out}}{\partial r} - \frac{\partial \phi_{in}}{\partial r} = -\frac{\sigma}{\varepsilon_0}$$

$$\sum_l -(l+1) \frac{B_l}{a^{l+2}} P_l - \sum_l l A_l r^{l-1} P_l = -\frac{\sigma_0}{\varepsilon_0} \cos(2\theta)$$

now $\cos(2\theta) = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1$

Recall $P_2(x) = \frac{3}{2} x^2 - \frac{1}{2}$

$P_0(x) = 1$

$$2x^2 - 1 = \frac{4}{3} P_2(x) - \frac{1}{3} P_0(x)$$

so, we have: $$\sum_l \left[\frac{-(l+1) B_l}{a^{l+2}} - l A_l a^{l-1} \right] P_l = \frac{-\sigma_0}{3\varepsilon_0} \left[4 P_2(x) - P_0(x) \right]$$

plug in (1) $$\sum_l \left[\frac{-(l+1) A_l a^{2l+1}}{a^{l+2}} - l A_l a^{l-1} \right] P_l$$

$$= \sum_l A_l a^{l-1} \left[\underbrace{-(l+1) - l}_{-2l-1} \right] P_l = -\frac{\sigma_0}{3\varepsilon_0} \left[4 P_2 - P_0 \right]$$

$A_l = 0 \quad l \neq 0,1$

$l = 0 \quad -A_0 a^{-1} = \frac{\sigma_0}{3\varepsilon_0}$

$l = 2 \quad A_2 a (-5) = -\frac{4}{3} \frac{\sigma_0}{\varepsilon_0}$

So $\quad A_0 = \frac{-\sigma_0}{3\varepsilon_0} a \qquad B_0 = A_0 a = \frac{-\sigma_0}{3\varepsilon_0} a^2$

$$A_2 = \frac{4\sigma_0}{15\varepsilon_0}\frac{1}{a} \qquad B_2 = A_0 a^5 = \frac{4\sigma_0}{15\varepsilon_0} a^4$$

$$\phi_{in} = -\frac{\sigma_0}{3\varepsilon_0} a + \frac{4}{15}\frac{\sigma_0}{\varepsilon_0}\frac{r^2}{a} P_2$$

$$= \frac{\sigma_0 a}{3\varepsilon_0}\left[-1 + \frac{4}{5}\frac{r^2}{a^2} P_2\right]$$

$$= \frac{\sigma_0 a}{3\varepsilon_0}\left[-1 + \frac{2}{5}\frac{r^2}{a^2}\left(3\cos^2\theta - 1\right)\right]$$

$$\phi_{out} = -\frac{\sigma_0}{3\varepsilon_0}\frac{a^2}{r} + \frac{4\sigma_0}{15\varepsilon_0}\frac{a^4}{r^3} P_2$$

$$= \frac{\sigma_0 a^2}{3\varepsilon_0 r}\left[-1 + \frac{4}{5}\frac{a^2 P_2}{r^2}\right]$$

$$= \frac{\sigma_0 a}{3\varepsilon_0}\frac{a}{r}\left[-1 + \frac{2}{5}\frac{a^2}{r^2}\left(3\cos^2\theta - 1\right)\right]$$

3. Method 2: Integration

$$\Phi = \frac{1}{4\pi\varepsilon_0}\int_0^\pi \frac{\sigma\, da}{|x-x'|}$$

$\sigma = \sigma_0 \cos 2\theta$

$= \sigma_0\left[\frac{4}{3}P_2 - \frac{1}{3}P_0\right]$

$$\int_0^\pi \frac{\sigma_0\left[\frac{4}{3}P_2 - \frac{1}{3}P_0\right]\cdot R^2 d\phi' \sin\theta' d\theta'}{|x-x'|}$$

on axis

$$= \frac{2\pi R^2\sigma_0}{4\pi\varepsilon_0}\sum_\ell \int_0^\pi \sin\theta' d\theta' \left[\frac{4}{3}P_2(\cos\theta') - \frac{1}{3}P_0(\cos\theta')\right]\frac{r_<^\ell}{r_>^{\ell+1}} P_\ell(\cos\theta')$$

$$\frac{1}{|x-x'|} = \sum_\ell \frac{r_<^\ell}{r_>^{\ell+1}} P_\ell(\cos\gamma)$$

on z axis $\cos\gamma \to \cos\theta'$

let $x = \cos\theta'$

$dx = -\sin\theta' d\theta'$

$$= \frac{\sigma_0 R^2}{2\varepsilon_0}\sum_\ell \frac{r_<^\ell}{r_>^{\ell+1}} \int_{-1}^1 dx\left[\frac{4}{3}P_2(x) - \frac{1}{3}P_0(x)\right] P_\ell(x)$$

$$\int P_\ell P_{\ell'} = \frac{2}{2\ell+1}\delta_{\ell\ell'}$$

$$= \frac{\sigma_0 R^2}{2\varepsilon_0}\sum_\ell \cancel{\frac{r_<^\ell}{r_>^{\ell+1}}}\left[\frac{4}{3}\cdot\frac{2}{(2(2)+1)}\frac{r_<^2}{r_>^3} - \frac{1}{3}\cdot\frac{2}{1}\frac{r_<^0}{r_>^1}\right]$$

$$= \frac{\sigma_0 R^2}{3\varepsilon_0}\left[-1\frac{r_<^0}{r_>^1} + \frac{4}{5}\frac{r_<^2}{r_>^3}\right]$$

since this is on axis, Multiply by appropriate P_ℓ

$$= \frac{\sigma_0 R^2}{3\varepsilon_0}\frac{1}{r_>}\left[-P_0 + \frac{4}{5}\frac{r_<^2}{r_>^2}P_2\right]$$

for $r < R$ $\quad \Phi_{in} = \frac{\sigma_0 R}{3\varepsilon_0}\left[-1 + \frac{4}{5}\frac{r^2}{R^2}P_2(\cos\theta)\right]$

$r > R$ $\quad \Phi_{out} = \frac{\sigma_0 R^2}{3\varepsilon_0 r}\left[-1 + \frac{4}{5}\frac{R^2}{r^2}P_2(\cos\theta)\right]$

4. method 1: separation of variables

$$\Phi(r,\phi) = \sum_{\nu} \left(A_\nu \cos(\nu\phi) + B_\nu \sin(\nu\phi)\right)\left(C_\nu r^\nu + \frac{D_\nu}{r^\nu}\right)$$

$$+ a_0 + b_0 \ln(r) + C_0 \phi$$

Note since we don't have $0 \le \phi \le 2\pi$ ν does not have to be an integer

we include $r=0$ $\Rightarrow b_0 = 0$ $D_\nu = 0$

$$\Phi(r,\phi) = \sum_{\nu} \left[A_\nu r^\nu \cos(\nu\phi) + B_\nu r^\nu \sin(\nu\phi)\right] + a_0 + C_0\phi$$

$$\Phi(r,\pi) = -V = \sum_{\nu} \left[A_\nu r^\nu \cos(\nu\pi) + B_\nu r^\nu \sin(\nu\pi)\right] + a_0 + C_0\pi$$

$$\Phi(r,0) = V = \sum_{\nu} \left[A_\nu r^\nu \cos(0) + B_\nu r^\nu \sin(0)\right] + a_0 + C_0(0)$$

(1) so, $\sum_{\nu} r^\nu \left[A_\nu\right] + a_0 = V$

(2) an $\sum_{\nu} r^\nu \left[A_\nu \cos(\nu\pi) + B_\nu \sin(\nu\pi)\right] + a_0 + C_0\pi = -V$

These can't be a function of r

so $A_\nu = 0$ from (1) and $a_0 = V$

$A_\nu \cos(\nu\pi) + B_\nu \sin(\nu\pi) = 0$ from 2

$B_\nu \sin(\nu\pi) = 0$ $B_\nu = 0$

or $\nu =$ integer

and $a_0 + C_0\pi = -V$

$V + C_0\pi = -V$ $C_0\pi = -2V$ $C_0 = -\frac{2V}{\pi}$

$$\Phi(r,\phi) = \underbrace{\sum_{\nu = \text{integer}} B_\nu \sin(\nu\phi) r^\nu}_{} + V - \frac{2V}{\pi}\phi$$

B_ν determined by boundary condition at ϕ

$\phi(r,\phi) = V\left(1 - \frac{2}{\pi}\phi\right)$ if no more conductors

4. Note: I think Seperation of Variables in cartesion coordinates would not work because the form of Φ is

$$\Phi = V_0\left(1 - \frac{2}{\pi}\phi\right) = V_0 - \frac{2V_0}{\pi}\tan^{-1}\frac{y}{x}$$

Now, I might be wrong but I dont think you can express this as $\Phi = X(x)\,Y(y)$.

The idea behind sep. of Var. is, you guess a solution that can be separated into functions of only one variable

$$\Phi = X(x)\,Y(y) \quad \text{or} \quad \Phi = R(\rho)\,F(\phi) \quad \text{etc}$$

and work from there.

If you can find a solution that satisfies boundary conditions, then you are guaranteed it is correct by the Uniqueness theorem. If not, then the form you started with does not work.

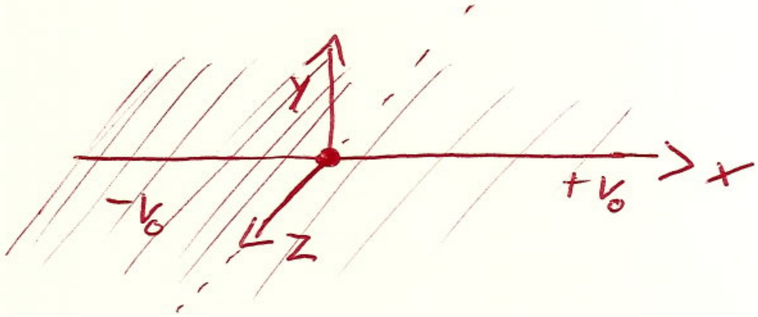
since $\tan^{-1}\frac{y}{x}$ is not a product of a function only of x and a function only of y, I dont think Sep of Var in cartesion cordinates works in this case.

4. Method 2 Green Functions

$G(\vec{x}, \vec{x}')$ for a plane in 3D is

$$G_{3D} = \frac{1}{\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{1/2}} - \frac{1}{\left[(x-x')^2+(y+y')^2+(z-z')^2\right]^{1/2}}$$

lets label our axes in the following way:



The conductor ~~plane~~ is on the xz plane at $y=0$

$$\Phi = \begin{cases} +V_0 & x>0 \\ -V_0 & x<0 \end{cases}$$

We can see that in the final form of Φ everywhere, there can be no dependance on z because of symmetry, which reflects the 2D nature of the problem. However, we cannot simply ignore the $(z-z')$ terms in G. We have to integrate them out.

Alternatively we can start with the 2D Greens Function, imagining 2 line charges above and below the plane: recall $\Phi_{\text{line charge}} = \frac{1}{2\pi\epsilon_0} \ln \frac{\vec{r}}{r_0}$

$$G_{2D} \sim \ln \frac{\vec{r}_+}{r_0} - \ln \frac{\vec{r}_-}{r_0}$$

and proceed from there.

Everyone started from the 3D case, so I will start there too remember $$\Phi = \underbrace{\frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\vec{x}')}{|x-x'|}\, d^3x}_{\rho=0 \text{ does not contribute}} + \frac{-1}{4\pi}\oint_S \Phi(x')\frac{\partial G}{\partial n'}\, da$$

here, n is $(-y)$

$$\frac{\partial G}{\partial n} = -\frac{\partial G}{\partial y'}\bigg|_{y'=0} = -\left[\frac{y-y'}{[(x-x')^2+(z-z')^2+(y-y')^2]^{3/2}} + \frac{y+y'}{[(x-x')^2+(z-z')^2+(y+y')^2]^{3/2}}\right]_{y'=0}$$

$$= \frac{-2y}{[(x-x')^2+(z-z')^2+y^2]^{3/2}}$$

$$\Phi = \frac{1}{4\pi}\int_{-\infty}^{\infty}\Phi(x')\,dx'\int_{-\infty}^{\infty}dz'\,\frac{2y}{[(x-x')^2+(z-z')^2+y^2]^{3/2}}$$

before we split the x' integral, lets calculate the z' integral

$$\int_{-\infty}^{\infty}\frac{dz'}{[(x-x')+(z-z')^2+y^2]^{3/2}} = \frac{2}{(x-x')^2+y^2}$$

see last page for details on doing integral

$$\Phi(x) = \frac{2y}{4\pi}\cdot 2\int_{-\infty}^{\infty}\Phi(x')\,\frac{dx'}{(x-x')^2+y^2}$$

$$= \frac{y}{\pi}V_0\left[\int_0^{\infty}\frac{dx'}{(x-x')^2+y^2} - \int_{-\infty}^{0}\frac{dx'}{(x-x')^2+y^2}\right]$$

$$\int\frac{dx'}{(x-x')^2+y^2} = -\frac{1}{y}\tan^{-1}\frac{x-x'}{y}$$

see last page for details again

$$= -\frac{V_0}{\pi}\left[\tan^{-1}(-\infty) - \tan^{-1}\frac{x}{y} - \tan^{-1}\frac{x}{y} + \tan^{-1}(\infty)\right]$$

$$\left[-2\tan^{-1}\left(\frac{x}{y}\right) + \tan^{-1}(-\infty) + \tan^{-1}(\infty)\right]$$

I'm not sure what branch to evaluate these on...

$$\tan^{-1}\infty = \frac{\pi}{2}(2n+1)$$

So I will write $\tan^{-1}(-\infty)+\tan^{-1}(\infty) = n\pi$ and determine n by boundary conditions

4

$$\Phi(x) = \frac{V_0}{\pi}\left[2\tan^{-1}\left(\frac{x}{y}\right) + n\pi \right]$$

Note: $\text{Arctan}\, X = \pi - \arctan\frac{1}{X}$

$$= \frac{V_0}{\pi}\left[\pi - 2\tan^{-1}\left(\frac{y}{x}\right) + n\pi \right] = V_0\left(n - \frac{2}{\pi}\tan^{-1}\frac{y}{x} \right)$$

to satisfy B.C, $n = 1$

$$= V_0\left[1 - \frac{2}{\pi}\tan^{-1}\frac{y}{x} \right]$$ recall $\tan^{-1}\frac{y}{x} = \phi$

which agrees with our previous method.

4

First Integral

$$\int_{-\infty}^{\infty} \frac{dz'}{\left[(x-x')^2+y^2+(z-z')^2\right]^{3/2}}$$

let $a^2 = (x-x')^2+y^2$

$$\frac{1}{a^3}\int_{-\infty}^{\infty} dz'\left[1+\left(\frac{z-z'}{a}\right)^2\right]^{-3/2}$$

$\tan u = \left(\frac{z-z'}{a}\right)$

$\frac{du}{\cos u} = -\frac{dz'}{a}$

$$-\frac{1}{a^2}\int_{+\frac{\pi}{2}}^{-\frac{\pi}{2}} \left[1+\tan^2 u\right]^{-3/2}\cos^{-2} u \, du$$

$$-\frac{1}{a^2}\int_{+\frac{\pi}{2}}^{-\pi/2} \cos u \, du = -\frac{1}{a^2}\sin u\Big|_{+\pi/2}^{-\pi/2} = \frac{2}{a^2} = \frac{2}{(x-x')^2+y^2}$$

Second Integral

$$\int \frac{dx'}{(x-x')^2+y^2} = \frac{1}{y^2}\int \frac{dx}{\left(1+\left(\frac{x-x'}{y}\right)^2\right)}$$

$\tan u = \frac{x-x'}{y}$

$\frac{du}{\cos^2 u} = -\frac{dx'}{y}$

$$= -\frac{1}{y}\int \frac{du}{\cos^2 u(1+\tan^2 u)} = -\frac{1}{y}\int du = -\frac{1}{y}\tan^{-1}\frac{x-x'}{y}$$

PHYSICS 210A, Winter 2010
Final Exam (100 points in total)

- Closed books and closed notes
- Please write down the necessary intermediate steps
- Write your answers in the space provided. Continue on the back if necessary.
- A formula sheet is provided in the last page of the exam.

Name: TA ID: ____________.

Problem 1: ____________

Problem 2: ____________

Problem 3: ____________

Problem 4: ____________

Problem 5: ____________

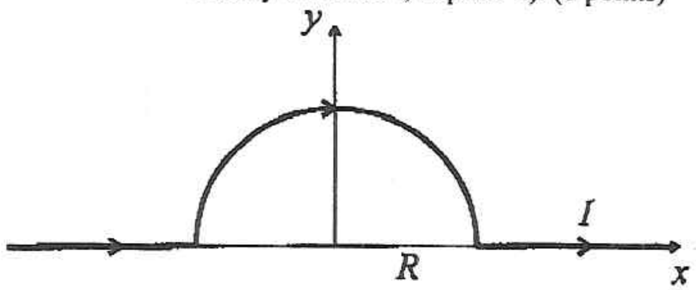
Problem 6: ____________

Total: ____________

1. A semicircular wire of radius R is centered at the origin, while straight segments extend to infinity along the X-axis, as shown below. A uniform current I is suddenly turned on at t = 0, remaining constant thereafter.

   (a) Calculate the vector ($\vec{A}$) and scalar potential (V) as a function of time at the origin. (10 points)

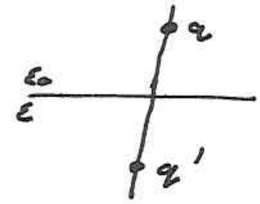
   (b) Calculate $\vec{E}$ and $\vec{B}$ as a function of time at the origin (if one of the quantities can not be directly calculated, explain it). (6 points)



see homework

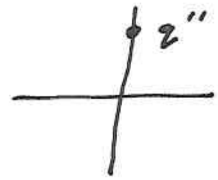
2. Suppose the entire region below the plane $z = 0$ is filled with uniform linear dielectric material of susceptibility χ_e. A point charge q is placed a distance d above the origin.

a) Find the potential in all space. (8 points)
b) Find the bound charge on the surface of the dielectric. (4 points)
c) Find the force acting on the charge q. (4 points)



for $z > 0$ imagine image charge q' at $z = -d$

for $z < 0$ imagine image charge q'' at $z = d$



a.

$$\Phi_{out} = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{(\rho^2 + (z-d)^2)^{1/2}} + \frac{q'}{(\rho^2 + (z+d)^2)^{1/2}}\right] \qquad \rho^2 = x^2 + y^2$$

$$\Phi_{in} = \frac{1}{4\pi\varepsilon_0}\frac{q''}{(\rho^2 + (z-d)^2)^{1/2}}$$

$$\vec{E}_{in} = \frac{q''}{4\pi\varepsilon_0}\left[\frac{\rho}{(\rho^2 + (z-d)^2)^{3/2}}\hat{\rho} + \frac{(z-d)}{(\rho^2 + (z-d)^2)^{3/2}}\hat{z}\right]$$

$$\vec{E}_{out} = \frac{q}{4\pi\varepsilon_0}\left[\frac{\rho}{(\rho^2 + (z-d)^2)^{3/2}} + \frac{q'}{q}\frac{\rho}{(\rho^2 + (z+d)^2)^{3/2}}\right]\hat{\rho}$$

$$+ \frac{q}{4\pi\varepsilon_0}\left[\frac{(z-d)}{(\rho^2 + (z-d)^2)^{3/2}} + \frac{q'}{q}\frac{(z+d)}{(\rho^2 + (z+d)^2)^{3/2}}\right]\hat{z}$$

$$\varepsilon/\varepsilon_0 = 1 + \chi_e$$

Apply B.C at $z = 0$

$E_{\parallel}$ cont: $\Rightarrow \quad 1 + \frac{q'}{q} = \frac{q''}{q}$

$D_{\perp}$ cont $\Rightarrow \quad 1 - \frac{q'}{q} = \frac{\varepsilon}{\varepsilon_0}\, q''/q$

$$q'/q = \frac{(1 - \varepsilon/\varepsilon_0)}{(1 + \varepsilon/\varepsilon_0)} = \frac{\chi_e}{2 + \chi_e}$$

$$q''/q = \frac{2}{(1 + \varepsilon/\varepsilon_0)} = \frac{2}{2 + \chi_e}$$

b. From Gauss's Law $\quad E_{out}^{z} - E_{in}^{z} = \sigma/\varepsilon_0$

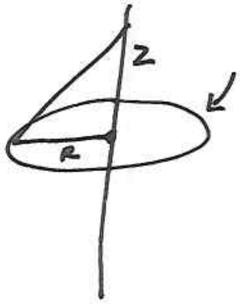
$$\sigma = \frac{qd}{4\pi}\frac{1}{(\rho^2 + d^2)^{3/2}}\left[-1 + \frac{q'}{q} + \frac{q''}{q}\right]$$

$$= \frac{qd}{2\pi}\frac{1 - \varepsilon/\varepsilon_0}{1 + \varepsilon/\varepsilon_0}\frac{1}{(\rho^2 + d^2)^{3/2}} = \frac{qd}{2\pi}\frac{\chi_e}{2 + \chi_e}\frac{1}{(\rho^2 + d^2)^{3/2}}$$

c.

$$F = \frac{q q'}{4\pi\varepsilon_0 (2d)^2} = \frac{q^2}{16\pi\varepsilon_0 d^2}\frac{\chi_e}{2 + \chi_e}$$

3. Consider a thin ring of radius R charged uniformly with a total charge Q. Find the potential at all points in space. [Hint: Write down the potential *on the axis* and match it to an expansion in Legendre polynomials. Try to do this for all terms in the series, but do it up to P_2 at least.] (16 points)



$$\lambda = Q/2\pi R \qquad \phi = \frac{\lambda}{4\pi\varepsilon_0}\int \frac{1}{\sqrt{R^2+z^2}}\, R\, d\phi$$

$$= \frac{Q}{4\pi\varepsilon_0}\frac{1}{\sqrt{R^2+z^2}} \quad \text{on } z \text{ axis}$$

General Solution $\quad \phi = \sum_\ell \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right) P_\ell(\cos\theta)$

on z axis $\to \quad = \sum_\ell (A_\ell r^\ell + B_\ell/r^{\ell+1})$

Expand $\frac{1}{\sqrt{R^2+z^2}}$ and match to

$$(1+\varepsilon^2)^{-1/2} = 1 + (-\tfrac{1}{2})\varepsilon^2 + \tfrac{1}{2}(-\tfrac{1}{2})(-\tfrac{3}{2})\varepsilon^4 + \tfrac{1}{2\cdot 3}(-\tfrac{1}{2})(-\tfrac{3}{2})(-\tfrac{5}{2})\varepsilon^6 + \dots$$
$$= \sum_j \frac{(-1)^j(2j-1)!!}{j!\,2^j}\varepsilon^{2j}$$

so for $z < R$

$$\phi_{\text{on axis}} = \frac{Q}{4\pi\varepsilon_0}\frac{1}{R}\sum_j \frac{(-1)^j(2j-1)!!}{j!\,2^j}\left(\frac{z}{R}\right)^{2j}$$

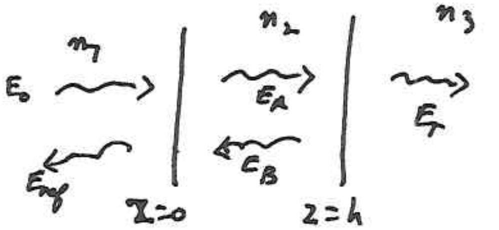
so
$$\phi_{\substack{\text{everywhere}\\ r<R}} = \frac{Q}{4\pi\varepsilon_0}\frac{1}{R}\sum_j \frac{(-1)^j(2j-1)!!}{j!\,2^j}\left(\frac{r}{R}\right)^{2j} P_{2j}(\cos\theta)$$

for $z > R$

$$\phi_{\text{on axis}} = \frac{Q}{4\pi\varepsilon_0}\frac{1}{z}\sum_j \frac{(-1)^j(2j-1)!!}{j!\,2^j}\left(\frac{R}{z}\right)^{2j}$$

$$\phi_{\substack{\text{everywhere}\\ r>R}} = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r}\sum_j \frac{(-1)^j(2j-1)!!}{j!\,2^j}\left(\frac{R}{r}\right)^{2j} P_{2j}(\cos\theta)$$

4. A beam of light is incident normally from air on a plane slab of a transparent dielectric with refractive index n_2 and thickness h. The light passes through the slab and enters a third medium with refractive index n_3 and of infinite extent. Find the condition for zero reflection back into the first medium. (16 points)



Fields: Region 1 $\quad E_x = E_0 e^{iK_1 z} + E_{ref} e^{-iK_1 z}$

$$B_y = \frac{E_0}{c} e^{iK_1 z} - \frac{E_{ref}}{c} e^{-iK_1 z}$$

2. $\quad E_x = E_A e^{iK_2 z} + E_B e^{-iK_2 z}$

$$B_y = \frac{n_2 E_A}{c} e^{iK_2 z} - \frac{n_2 E_B}{c} e^{-iK_2 z}$$

3. $\quad E_x = E_T e^{iK_3 z}$

$$B_y = \frac{n_3}{c} E_T e^{iK_3 z}$$

Apply B.C. $\quad E_{\parallel}$ cont. at $z=0, h$

$H_{\parallel}$ cont. at $z=0, h$

since μ is about the same everywhere, this is the same as $B_{\parallel}$ cont.

$$E_0 + E_{ref} = E_A + E_B$$

$$E_0 - E_{ref} = n_2 (E_A - E_B)$$

$$E_A e^{iK_2 h} + E_B e^{-iK_2 h} = E_T e^{iK_3 h}$$

$$E_A e^{iK_2 h} - E_B e^{-iK_2 h} = \frac{n_3}{n_2} E_T e^{iK_3 h}$$

Solve for $\frac{E_{ref}}{E_0}$: or leave E_{ref} in terms of E_T

$$E_A = \frac{E_T}{2} (1 + n_3/n_2) e^{iK_3 h} e^{-iK_2 h}$$

$$E_B = \frac{E_T}{2} (1 - n_3/n_2) e^{iK_3 h} e^{iK_2 h}$$

$$\frac{E_{ref}}{E_T} = \frac{e^{iK_3 h}}{4} \left[(1 - n_2)(1 + n_3/n_2) e^{-iK_2 h} + (1 + n_2)(1 - n_3/n_2) e^{iK_2 h} \right]$$

for zero reflection $\quad \left| \frac{E_{ref}}{E_T} \right| = 0$

$$(1 - n_2)(1 + n_3/n_2) + (1 + n_2)(1 - n_3/n_2) e^{2iK_2 h} = 0$$

Imaginary Part: $\quad 2K_2 h = m\pi \quad \Rightarrow \quad \boxed{h = \frac{m}{4} \lambda_2 = \frac{m}{4} \frac{\lambda_{vac}}{n_2}}$

$$K_2 = \frac{m\pi}{2h}$$

Real Part:

$$\boxed{\begin{array}{lll} m \text{ even:} & 2(1 - n_3) = 0 & \Rightarrow \quad n_3 = 1 \\ m \text{ odd} & 2(n_2^2 - n_3) = 0 & \Rightarrow \quad n_2 = \sqrt{n_3} \end{array}}$$

5. The linear charge density on a ring of radius a is given by $\rho = \frac{q}{a}(\cos\varphi - \sin\varphi)$. Find the monopole, dipole, and quadrupole moments of the system, and use these moments to write an expression for the potential at an arbitrary point in space. (16 points)

$$Q_{tot} = \frac{q}{a}\int(\cos\phi - \sin\phi)\, a\, d\phi = q\left[\int_0^{2\pi}\cos\phi\, d\phi + \int_0^{2\pi}\sin\phi\, d\phi\right] = 0$$

$$\vec{p} = \frac{q}{a}\int\begin{bmatrix}x\\y\\z\end{bmatrix}(\cos\phi - \sin\phi)\, a\, d\phi = q\int\begin{bmatrix}a\cos\phi\\a\sin\phi\\0\end{bmatrix}(\cos\phi - \sin\phi)\, d\phi$$

$$\int\cos^2\phi = \int\sin^2\phi = \pi$$

$$\int\sin\phi\cos\phi = 0$$

$$\boxed{\vec{p} = qa\pi\begin{bmatrix}1\\-1\\0\end{bmatrix}}$$

$$Q_{ij} = \int(3x_ix_j - r^2\delta_{ij})\,\rho(\vec{x})\, dV$$

$$Q_{zz} = \int\underbrace{(3z^2 - a^2)}_{\text{constant}}\, q/a\,\underbrace{(-\sin\phi + \cos\phi)}_{\text{full oscillation}} = 0 \qquad \text{same with } Q_{zy}, Q_{zx}$$

$$Q_{xx} = \int(3x^2 - a^2)\frac{q}{a}(\cos\phi - \sin\phi)\, a\, d\phi = qa^2\int(3\cos^2\phi - 1)(\cos\phi - \sin\phi)\, d\phi$$

$$3\int\underbrace{\cos^2\phi}_{\pi\ \text{periodicity}}\underbrace{(\cos\phi - \sin\phi)}_{2\pi\ \text{periodicity}} \qquad - \int(\cos\phi - \sin\phi)$$

$$= 0 \qquad\qquad = 0$$

same with Q_{yy}

$$Q_{xy} = \int(3xy)\frac{q}{a}(\cos\phi - \sin\phi)\, a\, d\phi = qa^2\int(3\underbrace{\sin\phi\cos\phi}_{\frac{1}{2}\sin 2\phi})(\cos\phi - \sin\phi)\, d\phi$$

$= 0$ for same reason as Q_{xx}

so $\boxed{Q_{ij} = 0}$

$$\text{so}\quad \phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2} = \frac{qa\pi}{4\pi\varepsilon_0}\left(\sin\theta\cos\phi + \sin\theta\sin\phi\right)$$

6. (a) X-rays which strike a metal surface at an angle of incidence to the normal greater than a critical angle are totally reflected. Assuming that the metal contains n free electrons per unit volume, calculate θ_c as a function of the angular frequency ω of X-rays, m_e, e, n and ε_0. (10 points)
(b) If ω and θ are such that total reflection does not occur, calculate what fraction of the incident wave is reflected. Assuming that the polarization vector of the X-rays is perpendicular to the plane of incidence and $\mu \approx \mu_0$. (10 points)

a. In general $\varepsilon_r = \frac{\varepsilon}{\varepsilon_0} = 1 + \frac{Ne^2}{m\varepsilon_0} \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j \omega}$

ω_j is the j^{th} binding frequency of each molecule
N is molecules/volume
f_j is # of electrons/molecule bound at frequency ω_j

in metals, we have free electrons in addition to some bound ones. For the free ones $\omega_0 = 0$, pull it out of the sum.

$$\varepsilon_r = 1 + \underbrace{\frac{Ne^2 f_0}{m\varepsilon_0(\omega_0^2 - \omega^2 - i\gamma_0\omega)}}_{\text{free electrons } \omega_0 = 0} + \underbrace{\sum_{j>0} \frac{Ne^2}{m\varepsilon_0} \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}}_{\text{rest of bound electrons}}$$

if there are no bound electrons,

$$\varepsilon_r = 1 + \frac{Ne^2 f_0}{m\varepsilon_0(-\omega^2 - i\gamma_0\omega)} = 1 + \frac{ne^2}{m\varepsilon_0} \frac{1}{(-\omega^2 - i\gamma_0\omega)}$$

where $n = Nf_0$ is the number of free electrons per volume.

and for high frequency (x-rays) $\omega \gg \gamma_0$

$$\varepsilon_r = 1 - \frac{\omega_p^2}{\omega^2} \quad \text{where} \quad \omega_p^2 = \frac{ne^2}{m\varepsilon_0}$$ is the plasma frequency of free electrons

$\omega_p \sim 10^{15}$ Hz ultraviolet.

Notice this is positive and less than 1 for high frequency, so we can get total reflection coming in from air.

$$\boxed{\sin\theta_c = \frac{\overset{\sqrt{\varepsilon_r}}{n'}}{\underset{=1}{n}} = \sqrt{\varepsilon_r} = \sqrt{1 - \omega_p^2/\omega^2}}$$

b. For polarization perpendicular to the plane of incidence, $\frac{E_{ref}}{E_{inc}} = \frac{\sin(\theta' - \theta)}{\sin(\theta' + \theta)}$

snells law: $\overset{=1}{n} \sin\theta = n' \sin\theta'$

$\sin\theta' = \frac{1}{n'}\sin\theta$

$\cos\theta' = \sqrt{1 - \frac{\sin^2\theta}{n'^2}}$

$\Rightarrow \sin(\theta' \pm \theta) = \sin\theta'\cos\theta \pm \cos\theta'\sin\theta = \frac{\sin\theta}{n'}\left(\cos\theta \pm \sqrt{n'^2 - \sin^2\theta}\right)$

so the Power fraction reflected is

$$\boxed{\left|\frac{E_{ref}}{E_i}\right|^2 = \left|\frac{\cos\theta - \sqrt{1 - \frac{\omega_p^2}{\omega^2} - \sin^2\theta}}{\cos\theta + \sqrt{1 - \frac{\omega_p^2}{\omega^2} - \sin^2\theta}}\right|^2}$$

PHYSICS 210A, Winter 2010
Final Exam (100 points in total)

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- A formula sheet is provided in the last page of the exam.

Name: TA ID: ______________.

Problem 1: ______________

Problem 2: ______________

Problem 3: ______________

Problem 4: ______________

Problem 5: ______________

Problem 6: ______________

Total: ______________

1. A semicircular wire of radius R is centered at the origin, while straight segments extend to infinity along the X-axis, as shown below. A uniform current I is suddenly turned on at t = 0, remaining constant thereafter.

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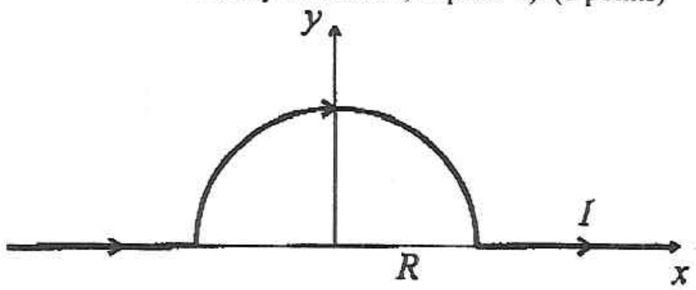


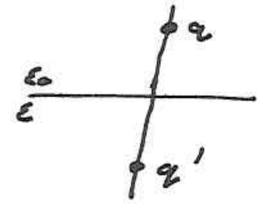
see homework

2. Suppose the entire region below the plane $z = 0$ is filled with uniform linear dielectric material of susceptibility χ_e. A point charge q is placed a distance d above the origin.

a) Find the potential in all space. (8 points)

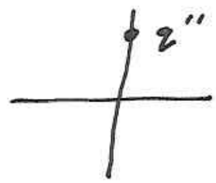
b) Find the bound charge on the surface of the dielectric. (4 points)

c) Find the force acting on the charge q. (4 points)



for $z > 0$ imagine image charge q' at $z = -d$

for $z < 0$ imagine image charge q'' at $z = d$



a.

$$\Phi_{out} = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{(\rho^2 + (z-d)^2)^{1/2}} + \frac{q'}{(\rho^2+(z+d)^2)^{1/2}}\right] \qquad \rho^2 = x^2+y^2$$

$$\Phi_{in} = \frac{1}{4\pi\varepsilon_0}\frac{q''}{(\rho^2+(z-d)^2)^{1/2}}$$

$$\vec{E}_{in} = \frac{q''}{4\pi\varepsilon_0}\left[\frac{\rho}{(\rho^2+(z-d)^2)^{3/2}}\hat{\rho} + \frac{(z-d)}{(\rho^2+(z-d)^2)^{3/2}}\hat{z}\right]$$

$$\vec{E}_{out} = \frac{q}{4\pi\varepsilon_0}\left[\frac{\rho}{(\rho^2+(z-d)^2)^{3/2}} + \frac{q'}{q}\frac{\rho}{(\rho^2+(z+d)^2)^{3/2}}\right]\hat{\rho}$$

$$+\frac{q}{4\pi\varepsilon_0}\left[\frac{(z-d)}{(\rho^2+(z-d)^2)^{3/2}} + \frac{q'}{q}\frac{(z+d)}{(\rho^2+(z+d)^2)^{3/2}}\right]\hat{z}$$

$$\varepsilon/\varepsilon_0 = 1+\chi_e$$

Apply B.C at $z=0$

$$E_{\parallel} \text{ cont:} \quad \Rightarrow \quad 1 + \frac{q'}{q} = \frac{q''}{q}$$

$$D_{\perp} \text{ cont} \quad \Rightarrow \quad 1 - \frac{q'}{q} = \frac{\varepsilon}{\varepsilon_0}\, q''/q$$

$$q'/q = \frac{(1-\varepsilon/\varepsilon_0)}{(1+\varepsilon/\varepsilon_0)} = \frac{\chi_e}{2+\chi_e}$$

$$q''/q = \frac{2}{(1+\varepsilon/\varepsilon_0)} = \frac{2}{2+\chi_e}$$

b. From Gauss's Law $\quad E_{out}^{z} - E_{in}^{z} = \sigma/\varepsilon_0$

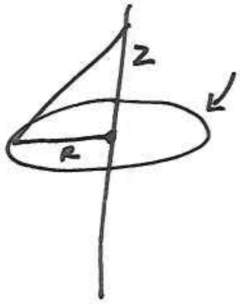
$$\sigma = \frac{qd}{4\pi}\frac{1}{(\rho^2+d^2)^{3/2}}\left[-1 + \frac{q'}{q} + \frac{q''}{q}\right]$$

$$= \frac{qd}{2\pi}\frac{1-\varepsilon/\varepsilon_0}{1+\varepsilon/\varepsilon_0}\frac{1}{(\rho^2+d^2)^{3/2}} = \frac{qd}{2\pi}\frac{\chi_e}{2+\chi_e}\frac{1}{(\rho^2+d^2)^{3/2}}$$

c.

$$F = \frac{qq'}{4\pi\varepsilon_0 (2d)^2} = \frac{q^2}{16\pi\varepsilon_0 d^2}\frac{\chi_e}{2+\chi_e}$$

3. Consider a thin ring of radius R charged uniformly with a total charge Q. Find the potential at all points in space. [Hint: Write down the potential *on the axis* and match it to an expansion in Legendre polynomials. Try to do this for all terms in the series, but do it up to P_2 at least.] (16 points)



$$\lambda = Q/2\pi R \qquad \phi = \frac{\lambda}{4\pi\varepsilon_0}\int \frac{1}{\sqrt{R^2+z^2}}\, R\, d\phi$$

$$= \frac{Q}{4\pi\varepsilon_0}\frac{1}{\sqrt{R^2+z^2}} \quad \text{on } z \text{ axis}$$

General Solution $\quad \phi = \sum_\ell \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right) P_\ell(\cos\theta)$

on z axis $\rightarrow \quad = \sum_\ell \left(A_\ell r^\ell + B_\ell/r^{\ell+1}\right)$

Expand $\frac{1}{\sqrt{R^2+z^2}}$ and match to

$$(1+\varepsilon^2)^{-1/2} = 1 + \left(-\tfrac{1}{2}\right)\varepsilon^2 + \tfrac{1}{2}\left(-\tfrac{1}{2}\right)\left(-\tfrac{3}{2}\right)\varepsilon^4 + \tfrac{1}{2\cdot 3}\left(-\tfrac{1}{2}\right)\left(-\tfrac{3}{2}\right)\left(-\tfrac{5}{2}\right)\varepsilon^6 + \ldots$$

$$= \sum_j \frac{(-1)^j (2j-1)!!}{j!\, 2^j}\varepsilon^{2j}$$

so for $z < R$

$$\phi_{\text{on axis}} = \frac{Q}{4\pi\varepsilon_0}\frac{1}{R}\sum_j \frac{(-1)^j (2j-1)!!}{j!\ 2^j}\left(\frac{z}{R}\right)^{2j}$$

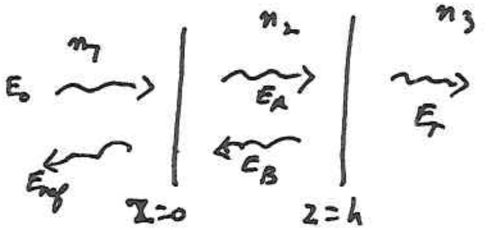
so $\phi_{\text{everywhere}}$ ($r < R$)
$$\phi_{\text{everywhere}} = \frac{Q}{4\pi\varepsilon_0}\frac{1}{R}\sum_j \frac{(-1)^j (2j-1)!!}{j!\ 2^j}\left(\frac{r}{R}\right)^{2j} P_{2j}(\cos\theta)$$

for $z > R$

$$\phi_{\text{on axis}} = \frac{Q}{4\pi\varepsilon_0}\frac{1}{z}\sum_j \frac{(-1)^j (2j-1)!!}{j!\ 2^j}\left(\frac{R}{z}\right)^{2j}$$

$\phi_{\text{everywhere}}$ ($r > R$)
$$\phi_{\text{everywhere}} = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r}\sum_j \frac{(-1)^j (2j-1)!!}{j!\ 2^j}\left(\frac{R}{r}\right)^{2j} P_{2j}(\cos\theta)$$

4. A beam of light is incident normally from air on a plane slab of a transparent dielectric with refractive index n_2 and thickness h. The light passes through the slab and enters a third medium with refractive index n_3 and of infinite extent. Find the condition for zero reflection back into the first medium. (16 points)



Fields: Region 1 $\quad E_x = E_0 e^{iK_1 z} + E_{ref} e^{-iK_1 z}$

$$B_y = \frac{E_0}{c} e^{iK_1 z} - \frac{E_{ref}}{c} e^{-iK_1 z}$$

2. $\quad E_x = E_A e^{iK_2 z} + E_B e^{-iK_2 z}$

$$B_y = \frac{n_2 E_A}{c} e^{iK_2 z} - \frac{n_2 E_B}{c} e^{-iK_2 z}$$

3. $\quad E_x = E_T e^{iK_3 z}$

$$B_y = \frac{n_3}{c} E_T e^{iK_3 z}$$

Apply B.C. $\quad E_{\parallel}$ cont. at $z=0, h$

$H_{\parallel}$ cont. at $z=0,h$

since μ is about the same everywhere, this is the same as $B_{\parallel}$ cont.

$$E_0 + E_{ref} = E_A + E_B$$

$$E_0 - E_{ref} = n_2 (E_A - E_B)$$

$$E_A e^{iK_2 h} + E_B e^{-iK_2 h} = E_T e^{iK_3 h}$$

$$E_A e^{iK_2 h} - E_B e^{-iK_2 h} = \frac{n_3}{n_2} E_T e^{iK_3 h}$$

Solve for $\frac{E_{ref}}{E_0}$: or leave E_{ref} in terms of E_T

$$E_A = \frac{E_T}{2}\left(1 + n_3/n_2\right) e^{iK_3 h} e^{-iK_2 h}$$

$$E_B = \frac{E_T}{2}\left(1 - n_3/n_2\right) e^{iK_3 h} e^{iK_2 h}$$

$$\frac{E_{ref}}{E_T} = \frac{e^{iK_3 h}}{4}\left[(1-n_2)(1+n_3/n_2) e^{-iK_2 h} + (1+n_2)(1-n_3/n_2) e^{iK_2 h}\right]$$

for zero reflection $\quad \left|\frac{E_{ref}}{E_T}\right| = 0$

$$(1-n_2)(1+n_3/n_2) + (1+n_2)(1-n_3/n_2) e^{2iK_2 h} = 0$$

Imaginary Part: $\quad 2K_2 h = m\pi \quad \Rightarrow \quad \boxed{h = \frac{m}{4}\lambda_2 = \frac{m}{4}\frac{\lambda_{vac}}{n_2}}$

$$K_2 = \frac{m\pi}{2h}$$

Real Part:

$$\boxed{\begin{array}{llll} m \text{ even:} & 2(1-n_3) = 0 & \Rightarrow & n_3 = 1 \\ m \text{ odd} & 2(n_2^2 - n_3) = 0 & \Rightarrow & n_2 = \sqrt{n_3} \end{array}}$$

5. The linear charge density on a ring of radius a is given by $\rho = \frac{q}{a}(\cos\varphi - \sin\varphi)$. Find the monopole, dipole, and quadrupole moments of the system, and use these moments to write an expression for the potential at an arbitrary point in space. (16 points)

$$Q_{tot} = \frac{q}{a}\int (\cos\phi - \sin\phi)\, a\, d\phi = q\left[\int_0^{2\pi} \cos\phi\, d\phi + \int_0^{2\pi} \sin\phi\, d\phi\right] = 0$$

$$\vec{p} = \frac{q}{a}\int \begin{bmatrix} x \\ y \\ z \end{bmatrix} (\cos\phi - \sin\phi)\, a\, d\phi = q\int \begin{bmatrix} a\cos\phi \\ a\sin\phi \\ 0 \end{bmatrix} (\cos\phi - \sin\phi)\, d\phi$$

$$\int \cos^2\phi = \int \sin^2\phi = \pi$$

$$\int \sin\phi\cos\phi = 0$$

$$\boxed{\vec{p} = qa\pi \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}$$

$$Q_{ij} = \int (3x_i x_j - r^2\delta_{ij})\, \rho(\vec{x})\, dV$$

$$Q_{zz} = \int \underbrace{(3z^2 - a^2)}_{\text{constant}}\, \tfrac{q}{a}\, \underbrace{(-\sin\phi + \cos\phi)}_{\text{full oscillation}} = 0 \qquad \text{same with } Q_{zy}, Q_{zx}$$

$$Q_{xx} = \int (3x^2 - a^2)\frac{q}{a}(\cos\phi - \sin\phi)\, a\, d\phi = qa^2\int (3\cos^2\phi - 1)(\cos\phi - \sin\phi)\, d\phi$$

$$3\int \underbrace{\cos^2\phi}_{\pi\text{ periodicity}}\, \underbrace{(\cos\phi - \sin\phi)}_{2\pi\text{ periodicity}} - \underbrace{\int (\cos\phi - \sin\phi)}_{=0}$$

$$= 0$$

same with Q_{yy}

$$Q_{xy} = \int (3xy)\frac{q}{a}(\cos\phi - \sin\phi)\, a\, d\phi = qa^2\int (3\underbrace{\sin\phi\cos\phi}_{\frac{1}{2}\sin 2\phi})(\cos\phi - \sin\phi)\, d\phi$$

$= 0$ for same reason as Q_{xx}

so $\boxed{Q_{ij} = 0}$

$$\text{so} \quad \phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2} = \frac{qa\pi}{4\pi\varepsilon_0}\left(\sin\theta\cos\phi + \sin\theta\sin\phi\right)$$

6. (a) X-rays which strike a metal surface at an angle of incidence to the normal greater than a critical angle are totally reflected. Assuming that the metal contains n free electrons per unit volume, calculate θ_c as a function of the angular frequency ω of X-rays, m_e, e, n and ε_0. (10 points)
(b) If ω and θ are such that total reflection does not occur, calculate what fraction of the incident wave is reflected. Assuming that the polarization vector of the X-rays is perpendicular to the plane of incidence and $\mu \approx \mu_0$. (10 points)

a. In general $\varepsilon_r = \frac{\varepsilon}{\varepsilon_0} = 1 + \frac{Ne^2}{m\varepsilon_0} \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j \omega}$

ω_j is the jth binding frequency of each molecule
N is molecules/volume
f_j is # of electrons/molecule bound at frequency ω_j

in metals, we have free electrons in addition to some bound ones. For the free ones $\omega_0 = 0$, pull it out of the sum.

$$\varepsilon_r = 1 + \underbrace{\frac{Ne^2 f_0}{m\varepsilon_0(\omega_0^2 - \omega^2 - i\gamma_0\omega)}}_{\substack{\text{free electrons} \\ \omega_0 = 0}} + \underbrace{\sum_{j>0} \frac{Ne^2}{m\varepsilon_0} \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}}_{\text{rest of bound electrons}}$$

if there are no bound electrons,

$$\varepsilon_r = 1 + \frac{Ne^2 f_0}{m\varepsilon_0(-\omega^2 - i\gamma_0\omega)} = 1 + \frac{ne^2}{m\varepsilon_0} \frac{1}{(-\omega^2 - i\gamma_0\omega)}$$

where $n = Nf_0$ is the number of free electrons per volume.

and for high frequency (x-rays) $\omega \gg \gamma_0$

$$\varepsilon_r = 1 - \frac{\omega_p^2}{\omega^2} \quad \text{where} \quad \omega_p^2 = \frac{ne^2}{m\varepsilon_0}$$

is the plasma frequency of free electrons

$\omega_p \sim 10^{15}$ Hz ultraviolet.

Notice this is positive and less than 1 for high frequency, so we can get total reflection coming in from air.

$$\boxed{\sin\theta_c = \frac{\overset{\sqrt{\varepsilon_r}}{n'}}{\underset{=1}{n}} = \sqrt{\varepsilon_r} = \sqrt{1 - \omega_p^2/\omega^2}}$$

b. For polarization perpendicular to the plane of incidence, $\frac{E_{ref}}{E_{inc}} = \frac{\sin(\theta' - \theta)}{\sin(\theta' + \theta)}$

snells law: $\overset{=1}{n}\sin\theta = n'\sin\theta'$

$\sin\theta' = \frac{1}{n'}\sin\theta$
$\cos\theta' = \sqrt{1 - \frac{\sin^2\theta}{n'^2}}$

$\Rightarrow \sin(\theta' \pm \theta) = \sin\theta'\cos\theta \pm \cos\theta'\sin\theta = \frac{\sin\theta}{n'}\left(\cos\theta \pm \sqrt{n'^2 - \sin^2\theta}\right)$

so the Power fraction reflected is

$$\boxed{\left|\frac{E_{ref}}{E_i}\right|^2 = \left|\frac{\cos\theta - \sqrt{1 - \frac{\omega_p^2}{\omega^2} - \sin^2\theta}}{\cos\theta + \sqrt{1 - \frac{\omega_p^2}{\omega} - \sin^2\theta}}\right|^2}$$

7

PHYSICS 210A, Fall 2010
Midterm Exam (100 points in total)

- Closed books and closed notes
- Please write down the necessary intermediate steps
- Write your answers in the space provided. Continue on the back if necessary.
- A formula sheet is provided at the end of the exam.

Name:____________________________ ID: ______________________________.

Problem 1: _________________________

Problem 2: _________________________

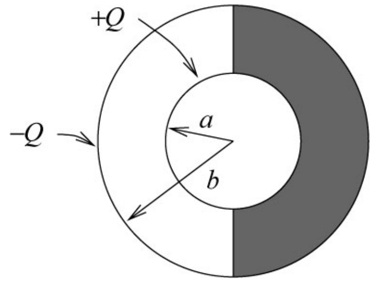
Problem 3: _________________________

Problem 4: _________________________

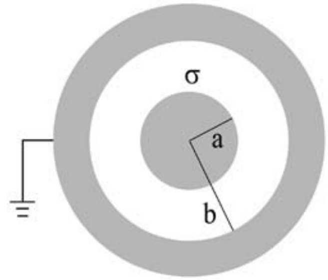
Total: _________________________

1. Two concentric conducting spheres of inner and outer radii a and b, respectively, carry charges $\pm Q$. The empty space between the spheres is half-filled by a hemispherical shell of dielectric (of dielectric constant $\varepsilon/\varepsilon_0$) as shown in the figure.

(a) Find the electric field everywhere between the spheres. (12 points)
(b) Calculate the surface-charge distribution on the inner sphere (free charge only). (7 +points)
(c) Calculate the bound (or polarization) charge density induced on the surface of the dielectric at $r = a$. (6 points)



2. An infinitely long cylinder of radius a and surface charge density $\sigma = \sigma_0 \cos 3\varphi$ is surrounded by an infinitely long **conducting** cylindrical tube of inner radius b which is held at zero potential.



(a) Find the potential $\Phi(r,\varphi)$ in the $0 \le r < a$ and the $a < r \le b$ regions. (20)

(b) Find the surface charge density on the inner surface of the grounded cylindrical tube. (10 points)

3. A sphere of radius R_1 has a charge density ρ uniform within its volume, except for a small spherical hollow region of radius R_2 located a distance a from the center. Find the electric field everywhere in the hollow sphere. (15)

4. An electric dipole of moment $\vec{p}$ is placed at a distance d from a grounded conducting sphere of radius a. The dipole is oriented in the direction radially away from the sphere. Assume that $d >> a$. Find the electrostatic potential outside the sphere. (30 points)

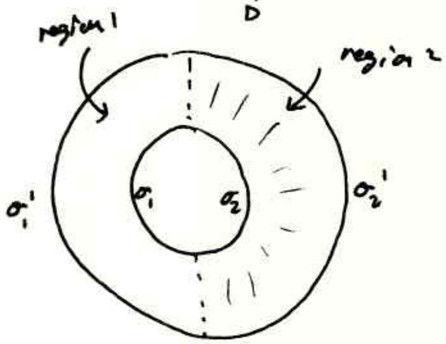
Midterm Solutions

1. For full solution see Homework.

Briefly,

$$\nabla \cdot E = \rho/\varepsilon_0 = \frac{1}{\varepsilon_0}(\rho_f + \rho_b) \qquad \rho_b = -\nabla \cdot P$$

$$\nabla \cdot (\underbrace{\varepsilon_0 E + P}_{D}) = \rho_f \qquad \nabla \cdot D = \rho_f$$



Gauss's Law on hemisphere

$$\frac{4\pi r^2}{2} \cdot D = \frac{4\pi a^2}{2} \sigma_2$$

$$\Rightarrow D_1 = \frac{a^2}{r^2}\sigma_1 = \varepsilon_0 E$$

$$D_2 = \frac{a^2}{r^2}\sigma_2 = \varepsilon E$$

$E_{\parallel}$ is continuous (radial) $\Rightarrow$ $\hat{r} \cdot \vec{E}_1 = \hat{r}_2 \cdot \vec{E}_2$ $\Rightarrow$ $\frac{\sigma_2}{\varepsilon} = \frac{\sigma_1}{\varepsilon_0}$ $\boxed{\sigma_2 = \frac{\varepsilon}{\varepsilon_0}\sigma_1}$

$$Q_2' = -Q_2 \Rightarrow \frac{4\pi}{2}(a^2 \sigma_2) = -\frac{4\pi}{2} b^2 \sigma_2'$$

$$\boxed{\begin{aligned} \sigma_2' &= -\frac{a^2}{b^2}\sigma_2 \\ \sigma_1' &= -\frac{a^2}{b^2}\sigma_1 \end{aligned}}$$

enforcing a total of Q on each sphere,

$$\frac{4\pi a^2}{2}(\sigma_1 + \sigma_2) = Q \qquad \sigma_1 + \sigma_2 = \frac{Q}{2\pi a^2}$$

Pt b

$$\boxed{\sigma_1 = \frac{Q}{2\pi a^2}\frac{1}{1+\varepsilon/\varepsilon_0} \qquad \sigma_2 = \frac{Q}{2\pi a^2}\frac{\varepsilon/\varepsilon_0}{1+\varepsilon/\varepsilon_0}}$$

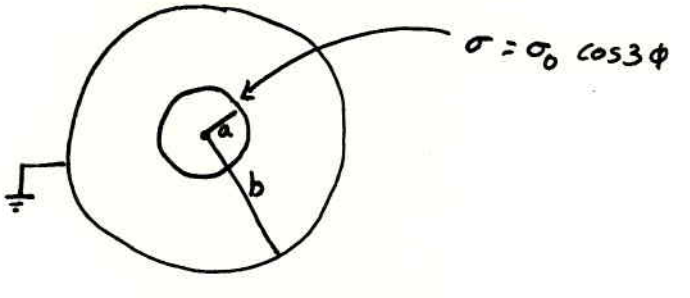
Pt a:

$$\boxed{\vec{E}_1 = \vec{E}_2 = \frac{a^2}{r^2}\frac{\sigma_1}{\varepsilon_0} = \frac{Q}{2\pi\varepsilon_0}\frac{1}{1+\varepsilon/\varepsilon_0}\frac{1}{r^2}}$$

c. the bound charge shields part of the free charge to make the E field constant.

$$\sigma_b + \sigma_2 = \sigma_1 \Rightarrow \boxed{\sigma_b = \sigma_1 - \sigma_2 = \frac{Q}{2\pi a^2}\frac{1-\varepsilon/\varepsilon_0}{1+\varepsilon/\varepsilon_0}}$$

2.



a. Use separation of variables in polar coordinates.

$r < a \qquad \phi_{in} = a_0 + \sum_{n=1}^{\infty} r^n (A_n \sin n\phi + B_n \cos n\phi)$

$b > r > a \qquad \phi_{out} = d_0 + b_0 \ln r + \sum_{n=1}^{\infty} \left[G_n r^n + H_n / r^n \right] \left[D_n \sin n\phi + F_n \cos n\phi \right]$

From the form of σ, we can guess only $n=3$ will survive but I will leave everything for now.

Apply Boundary Conditions:

- At $r=b$ $\phi = 0$, there can be no ϕ dependence

$$0 = d_0 + b_0 \ln b + \sum_n \left[G_n b^n + \frac{H_n}{b^n} \right] \left[D_n \sin n\phi + F_n \cos n\phi \right]$$

so $d_0 = -b_0 \ln b$ and $H_n = -G_n b^{2n}$ -> absorbe G_n into D_n, F_n

notice, saying $D_n = F_n = 0$ would be wrong in this case. Although it does satisfy $\phi = 0$ at $r=b$, we will loose the ϕ dependence at $r \neq b$, and would not be able to match ϕ at $r=a$.

Apply

- $\frac{\partial \phi_{in}}{\partial r} - \frac{\partial \phi_{out}}{\partial r} = +\frac{\sigma_\phi}{\epsilon_0}$

$$\frac{\partial \phi_{in}}{\partial r} = \sum_n n r^{n-1} (A_n \sin n\phi + B_n \cos n\phi)$$

$$\frac{\partial \phi_{out}}{\partial r} = \frac{b_0}{r} + \sum_n n r^{n-1} \left[1 - \frac{b^{2n}}{r^{2n}} \right] \left[D_n \sin n\phi + F_n \cos n\phi \right]$$

$$\Rightarrow \quad +\frac{\sigma_0}{\epsilon_0} \cos 3\phi = -\frac{b_0}{a} + \sum_n n a^{n-1} \left[\left(A_n - \left(1 + \left(\tfrac{b}{a}\right)^{2n}\right) D_n \right) \sin n\phi + \left(B_n - F_n \left(1 + \left(\tfrac{b}{a}\right)^{2n}\right) \right) \cos n\phi \right]$$

so $b_0 = 0$, $n = 3$, $\left(A_n - 1 + \left(\frac{b}{a}\right)^{2n} D_n\right) = 0$, $B_n - F_n \left(1 + \left(\frac{b}{a}\right)^{2n}\right) = +\frac{\sigma_0}{3 \epsilon_0 a^2}$

$$\Phi_{in} = a_0 + r^3\left[D_3 \left(1+ \left(\tfrac{b}{a}\right)^6\right) \sin 3\phi + \left(F_n \left(1+\left(\tfrac{b}{a}\right)^6\right) + \frac{\sigma_0}{3\varepsilon_0 a^2}\right) \cos 3\phi\right]$$

$$\Phi_{out} = r^3\left(D_3 \sin 3\phi + F_3 \cos 3\phi\right)\left(1 - \frac{b^6}{r^6}\right)$$

Apply $\Phi_{in} = \Phi_{out}$ at $r=a$

$$a_0 + a^3 D_3 \left(1+\left(\tfrac{b}{a}\right)^6\right)\sin 3\phi + a^3\left[F_3\left(1+\left(\tfrac{b}{a}\right)^6\right) + \frac{\sigma_0}{3\varepsilon_0 a^2}\right]\cos 3\phi = a^3\left(1-\left(\tfrac{b}{a}\right)^6\right)\left(D_3 \sin 3\phi + F_3 \cos 3\phi\right)$$

$a_0 = 0$

$a^3 D_3 \left(1+\left(\tfrac{b}{a}\right)^6\right)\sin 3\phi$ can never equal $a^3 D_3 \left(1-\left(\tfrac{b}{a}\right)^6\right)\sin 3\phi$ so $D_3 = 0$

$$\Rightarrow \left(1-\left(\tfrac{b}{a}\right)^6\right) F_3 = F_3\left(1+\left(\tfrac{b}{a}\right)^6\right) + \frac{\sigma}{3\varepsilon_0 a^2} \qquad \Rightarrow F_3 = \frac{-\sigma_0}{6\varepsilon_0 a^2} \frac{1}{(b/a)^6}$$

Finally,

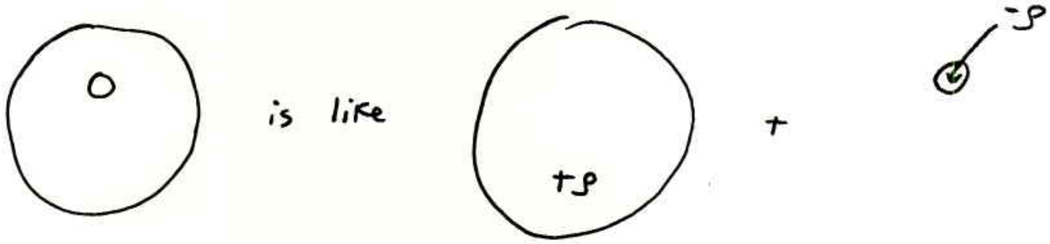
$$\boxed{\begin{aligned} \Phi_{in} &= \frac{+\sigma_0}{6\varepsilon_0}\left(1-\left(\tfrac{a}{b}\right)^6\right)\frac{r^3}{a^2}\cos 3\phi \\ \Phi_{out} &= \frac{+\sigma_0}{6\varepsilon_0}\left(1-\frac{r^6}{b^6}\right)\frac{a^4}{r^3}\cos 3\phi \end{aligned}}$$

b.

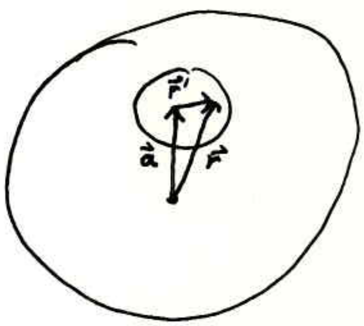
$$\frac{\sigma_f}{\varepsilon_0} = \left.\frac{\partial \Phi_{out}}{\partial r}\right|_{r=b} = \frac{\sigma_0}{6\varepsilon_0}\frac{a^4}{b^6}\cos 3\phi\left.\left(3r^2 + 3b^6/r^4\right)\right|_{r=b}$$

$$\boxed{\sigma_f = -\sigma_0 \frac{a^4}{b^4}\cos 3\phi}$$

3. Use Superposition



From Gauss's Law $\quad 4\pi r^2 E = \frac{\rho}{\varepsilon_0}\frac{4}{3}\pi r^3 \Rightarrow E = \frac{\rho}{3\varepsilon_0} r$ radially from center of sphere



$$\vec{E}_\rho = \frac{\rho}{3\varepsilon_0}\vec{r}$$

$$\vec{E}_{-\rho} = \frac{-\rho}{3\varepsilon_0}\vec{r}'$$

$$\vec{E}_{TOT} = \vec{E}_\rho + \vec{E}_{-\rho} = \frac{\rho}{3\varepsilon_0}(\vec{r} - \vec{r}')$$

Notice from the figure $\vec{r} = \vec{a} + \vec{r}'$ where $\vec{a} = a\hat{z}$

so $\vec{r}' = \vec{r} - \vec{a}$

Thus $\vec{E} = \frac{\rho}{3\varepsilon_0}\vec{a}$ the field is uniform in the hole

4. I will use the method of Images, but I do not remember what the image charge for a sphere looks like, so I will derive it,

$$\Phi = \sum_\ell \frac{A_\ell}{r^{\ell+1}} P_\ell \quad + \frac{q}{4\pi\varepsilon_0}\sum_\ell \frac{1}{d}\left(\frac{r}{d}\right)^\ell P_\ell$$

$= 0$ at $r = a$

$$0 = \sum_\ell \left(\frac{A_\ell}{a^{\ell+1}} + \frac{a^\ell}{d^{\ell+1}}\right) P_\ell(\cos\theta) \cdot \frac{q}{4\pi\varepsilon_0}$$

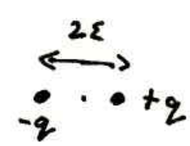
$$A_\ell = -\frac{a^{2\ell+1}}{d^{\ell+1}}$$

so
$$\Phi = \frac{q}{4\pi\varepsilon_0}\left[\sum_\ell \frac{r^\ell}{d^{\ell+1}} P_\ell + \underbrace{\sum_\ell \frac{-a^{2\ell+1}}{d^{\ell+1}} \frac{1}{r^{\ell+1}} P_\ell}_{\text{Write as } -\frac{a}{d}\sum_\ell \frac{\left(\frac{a^2}{d}\right)^\ell}{r^{\ell+1}} P_\ell}\right]$$

thus, the image charge is $q' = -\frac{a}{d} q$

at $d' = \frac{a^2}{d}$

Back to the dipole problem. Model the dipole as a positive and negative charge a distance 2ε apart.
where $p = 2\varepsilon q$



we have charge q at $d+\varepsilon$, and $-q$ at $d-\varepsilon$
The images are

$$-\frac{a}{d+\varepsilon} q \quad \text{at} \quad \frac{a^2}{d+\varepsilon}$$

and
$$\frac{a}{d-\varepsilon} q \quad \text{at} \quad \frac{a^2}{d-\varepsilon}$$

So, inside the sphere we have the image charges:

$r=0$ $\qquad -\frac{aq}{d+\varepsilon} \qquad \frac{aq}{d-\varepsilon}$

$\underbrace{\qquad\qquad}_{\frac{a^2}{d+\varepsilon}}$

$\underbrace{\qquad\qquad\qquad}_{\frac{a^2}{d-\varepsilon}}$

Since the charges are of different magnitude, this is NOT a pure dipole, it is a dipole + a monopole.

I will extract out the dipole and monopole terms

Image dipole: separate $+\frac{aq}{d+\varepsilon}$ of the $\frac{aq}{d-\varepsilon}$ out:

$$\frac{aq}{d-\varepsilon} = \frac{aq}{d+\varepsilon} - \frac{aq}{d+\varepsilon} + \frac{aq}{d-\varepsilon} = \frac{aq}{d+\varepsilon} + aq\left(-\frac{1}{d+\varepsilon} + \frac{1}{d-\varepsilon}\right)$$

$$= \underbrace{\frac{aq}{d+\varepsilon}}_{\text{contributes to dipole}} + \underbrace{\frac{aq}{d^2-\varepsilon^2}\cdot 2\varepsilon}_{\text{monopole term}}$$

We have

$-\frac{a}{d+\varepsilon}q \qquad \frac{aq}{d+\varepsilon} + \frac{aq}{d^2-\varepsilon^2}\cdot 2\varepsilon$

$$\underbrace{\qquad\qquad}_{a^2\left(\frac{1}{d-\varepsilon} - \frac{1}{d+\varepsilon}\right) = \frac{a^2\cdot 2\varepsilon}{d^2-\varepsilon^2}}$$

dipole term: $p' =$ distance $\cdot$ charge

$$= \frac{a^2\cdot 2\varepsilon}{d^2-\varepsilon^2}\cdot\frac{aq}{d+\varepsilon}$$

$$\simeq \frac{a^3}{d^3}\cdot 2\varepsilon q = \frac{a^3}{d^3}p$$

So the image dipole is a factor of $\frac{a^3}{d^3}$ smaller than the actual dipole at a distance of $\frac{a^2}{d}$ from the center of the sphere.

The monopole term is $\frac{aq\cdot 2\varepsilon}{d^2-\varepsilon^2} = \frac{a}{d^2}p$

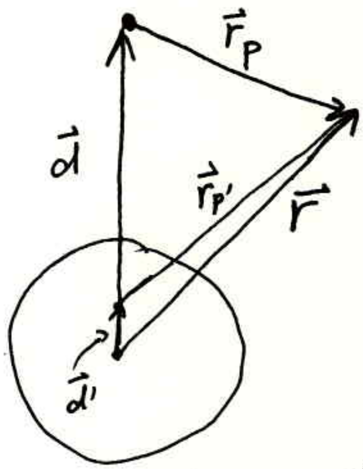
at a distance of $\frac{a^2}{d}$ as well.

$$p' = \frac{a^3}{d^3} p \qquad d' = \frac{a^2}{d}$$

Summing up

$$\Phi_{TOTAL} = \frac{1}{4\pi\varepsilon_0}\left[\underbrace{\frac{\vec{p}\cdot\vec{r}_p}{r_p^{\,3}}}_{\text{dipole}} + \underbrace{\frac{\vec{p}\,'\cdot\vec{r}_{p'}}{r_{p'}^{\,3}}}_{\text{image dipole}} + \underbrace{\frac{\frac{a}{d^2}p}{r_p}}_{\text{image monopole}}\right]$$

now we need to figure out what $\vec{r}_p$, $\vec{r}_{p'}$ are



$$\vec{d} = d\hat{z} \qquad \vec{r} = \vec{d} + \vec{r}_p \Rightarrow \vec{r}_p = \vec{r} - \vec{d}$$

$$\vec{d}' = d'\hat{z} \qquad \vec{r} = \vec{d}' + \vec{r}_{p'} \qquad \vec{r}_{p'} = \vec{r} - \vec{d}'$$

and $|r_p|^2 = r^2 + d^2 - 2rd\cos\theta$

$|r_p'|^2 = r^2 + d'^2 - 2rd'\cos\theta$

so,

$$\Phi_{TOTAL} = \frac{p}{4\pi\varepsilon_0}\left[\frac{\hat{z}\cdot(\vec{r}-\vec{d})}{(r^2+d^2-2rd\cos\theta)^{3/2}} + \frac{\frac{a^3}{d^3}\,\hat{z}\cdot(\vec{r}-\vec{d}')}{(r^2+d'^2-2rd'\cos\theta)^{3/2}} + \frac{\frac{a}{d^2}}{(r^2+d'^2-2rd'\cos\theta)^{1/2}}\right]$$

$$= \frac{p}{4\pi\varepsilon_0}\left[\frac{r\cos\theta - d}{(r^2+d^2-2rd\cos\theta)^{3/2}} + \frac{a^3}{d^3}\frac{\left(r\cos\theta - \frac{a^2}{d}\right)}{\left(r^2+\left(\frac{a^2}{d}\right)^2 - 2r\frac{a^2}{d}\cos\theta\right)^{3/2}} + \frac{a}{d^2}\frac{1}{\left(r^2+\left(\frac{a^2}{d}\right)^2-2r\frac{a^2}{d}\cos\theta\right)^{1/2}}\right]$$

you may verify that at $r=a$, $\phi = 0$ for all θ

PHYSICS 210B, Winter 2010
Final Exam (100 points in total)

- Closed books and closed notes
- Please write down the necessary intermediate steps
- Write your answers in the space provided. Continue on the back if necessary.
- A formula sheet is provided at the end of the exam.

Name:__________________________ ID: ____________________________.

Problem 1: _______________________

Problem 2: _______________________

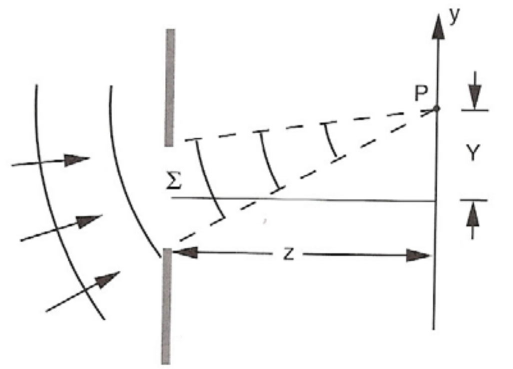
Problem 3: _______________________

Problem 4: _______________________

Problem 5: _______________________

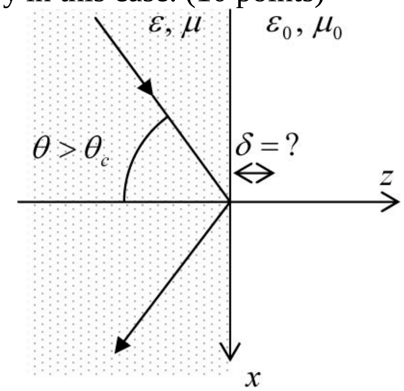
Total: _______________________

1. An aperture Σ in an opaque screen is illuminated by a spherical wave converging towards a point P located in a parallel plane a distance *z* behind the screen (shown below).
    (a) Find the quadratic-phase approximation to the illuminating wavefront in the plane of the aperture, assuming that the coordinates of P in the (x, y) plane are (0, Y). (10 points)
    (b) Assuming Fresnel diffraction from the plane of the aperture to the plane containing P, show that in the above case the observed intensity distribution is the Fraunhofer diffraction pattern of the aperture, centered on the point P. (10 points)



2. Two point charges $+q$ and $-q$ are placed at opposite poles of a spherical balloon of initial radius R_0. The radius of the balloon is set to oscillate as follows: $R(t) = R_0 + \rho \sin \omega t$ where $\rho\omega << c$.
   (a) Determine the total power radiated by the oscillating balloon, if any, in term of q, R_0, ρ and ω. (10 points)
   (b) Suppose instead that charges are deposited on the balloon as described below. For each case, determine the ratio of the total power radiated by the oscillating balloon, if any, to the total power radiated in (a). Show your work and explain your reasoning. (10 points)
      - One point charge $+q$ is placed at a given point on the balloon. The radius of the balloon is set to oscillate as above.
      - Total charge $+q$ is deposited uniformly on the surface of the balloon. The radius of the balloon is set to oscillate as above.

3. An electromagnetic wave, $\vec{E} = E_0 e^{i(kx\sin\theta + kz\cos\theta - \omega t)}\hat{y}$, is incident from inside a media with $\varepsilon\mu > \varepsilon_0\mu_0$ on its plane surface as shown in the figure below, where the incident angle (θ) is larger than the critical angle (θ_c).
   (a) Determine the depth of penetration (δ) of the evanescent wave into the free space in term of θ, θ_c, c and λ. Does the result depend on the wave polarization? [Hint: δ is defined as the depth when the magnitude of the evanescent wave decreases to $1/e$.] (10 points)
   (b) Show that there is no energy transport across the boundary in this case. (10 points)



4. A relativistic particle with the rest mass *m* and energy total *E* collides with a similar particle, initially at rest in the laboratory frame. Find:

(a) The velocity of the center of mass of the system in the lab frame. (7 points)

(b) The total energy of the system in the center-of-mass frame. (7 points)

(c) The final velocities of both particles (in the lab frame), if their final velocities are parallel to the incoming velocity. (6 points)

5. A relativistic particle of charge q is constrained to move along a circle of radius a at a constant angular frequency ω. The circle lies on the x-y plane of a Cartesian coordinate system which has an origin that is at the center of the circle.
   (a) Find the retarded time t' associated with an observation made at time t in the lab frame at point b along the z-axis (perpendicular to the circle). (5 point)
   (b) Find the scalar potential (Φ) measured at this point at time t in the lab frame. (8 points)
   (c) Find the vector potential ($\vec{A}$) measured at the same location and time t. (7 points)

Formula Sheet for Final

You may use any of the following equations without derivation.

$$\vec{E} = \vec{E}_0 e^{-i(\omega t - \vec{k}\cdot\vec{r})} \qquad \vec{B} = \vec{B}_0 e^{-i(\omega t - \vec{k}\cdot\vec{r})} \qquad \vec{B} = \sqrt{\mu\varepsilon}\,\hat{k}\times\vec{E}$$

$$v = \frac{\omega}{k} = \frac{1}{\sqrt{\varepsilon\mu}} \qquad n = \frac{\sqrt{\mu\varepsilon}}{\sqrt{\mu_0\varepsilon_0}} \qquad k = \frac{2\pi}{\lambda} \quad \theta_c = \sin^{-1}\left(\frac{n'}{n}\right)$$

$$\text{TE waves: } \vec{\nabla}_t B_z = -\frac{ik_c^2}{k_g}\vec{B}_t \qquad \vec{B}_t = \frac{k_g}{\omega}\left(\hat{z}\times\vec{E}_t\right) \quad k_0^2 = k_c^2 + k_g^2$$

$$\text{TM waves: } \vec{\nabla}_t E_z = -\frac{ik_c^2}{k_g}\vec{E}_t \qquad \vec{E}_t = -\frac{k_g}{\mu\varepsilon\omega}\left(\hat{z}\times\vec{B}_t\right)$$

$$\vec{p} = \int \vec{x}'\rho(\vec{x}')d^3x' \qquad \vec{A}_{ED}(\vec{x}) = -\frac{i\mu_0\omega\vec{p}}{4\pi}\frac{e^{ikr}}{r} \qquad \vec{B}_{ED} = \frac{\mu_0 ck^2}{4\pi}\frac{e^{ikr}}{r}\hat{n}\times\vec{p}$$

$$\vec{E}_{ED} = -\frac{k^2}{4\pi\varepsilon_0}\frac{e^{ikr}}{r}\left[\hat{n}\times(\hat{n}\times\vec{p})\right] = Z_0\vec{H}_{ED}\times\hat{n} \qquad Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \qquad \frac{dP}{d\Omega} = \frac{c^2k^4Z_0}{32\pi^2}\left|\hat{n}\times\vec{p}\right|^2$$

$$\vec{m} = \frac{1}{2}\int(\vec{x}'\times\vec{J})d^3x' \quad \vec{B}_{MD} = \frac{\mu_0k^2}{4\pi}\frac{e^{ikr}}{r}(\hat{n}\times\vec{m})\times\hat{n} \qquad \vec{E}_{MD} = \frac{Z_0k^2}{4\pi}\frac{e^{ikr}}{r}\hat{n}\times\vec{m} \qquad \frac{dP}{d\Omega} = \frac{k^4Z_0}{32\pi^2}\left|\hat{n}\times\vec{m}\right|^2$$

$$\frac{d\sigma}{d\Omega} = \frac{k^4a^6}{2}\left|\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right|^2\left(1+\cos^2\theta\right) \qquad \frac{d\sigma}{d\Omega} = k^4a^6\left[\frac{5}{8}\left(1+\cos^2\theta\right) - \cos\theta\right]$$

$$\frac{d\sigma}{d\Omega} = \frac{k^4a^6}{2}\left|\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right|^2\left(1+\cos^2\theta\right)\left|S(\vec{q})\right|^2 \qquad S(\vec{q}) = \sum_i e^{-i\vec{q}\cdot\vec{x}_i}$$

$$\frac{d\sigma}{d\Omega} = \frac{1+\cos^2\theta}{2}r_0^2 \qquad \sigma = \frac{8\pi}{3}r_0^2$$

$$\begin{pmatrix} x'_0 \\ x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad u'_{\parallel} = \frac{u_{\parallel} - v}{1 - \vec{v}\cdot\vec{u}/c^2} \qquad \vec{u}'_{\perp} = \frac{\vec{u}_{\perp}}{\gamma(1-\vec{v}\cdot\vec{u}/c^2)}$$

$$\vec{p} = \gamma m_0\vec{u} \qquad E = \gamma m_0 c^2 \qquad p^{\mu} = (E/c, \vec{p}) \qquad E = \sqrt{m_0^2c^4 + c^2p^2}$$

$$\partial_{\mu}G^{\mu\nu} = \frac{4\pi}{c}J^{\nu} \qquad \partial_{\mu}\wp^{\mu\nu} = 0 \qquad F^{\mu\nu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}$$

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c^3}\left|\dot{\vec{v}}\right|^2\sin^2\theta \qquad P = \frac{2}{3}\frac{q^2}{c^3}\left|\dot{\vec{v}}\right|^2$$

$$\frac{dP(t')}{d\Omega} = \frac{q^2\dot{v}^2}{4\pi c^3}\frac{\sin^2\theta}{(1-\beta\cos\theta)^5} \qquad P = \frac{2}{3}\frac{q^2}{m^2c^3}\left(\frac{dp}{dt}\right)^2$$

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c^3}\frac{\dot{v}^2}{(1-\beta\cos\theta)^3}\left[1 - \frac{\sin^2\theta\cos^2\phi}{\gamma^2(1-\beta\cos\theta)^2}\right] \qquad P = \frac{2}{3}\frac{q^2c}{\rho^2}\beta^4\gamma^4$$

$$\Phi(\vec{x},t) = \frac{q}{(1-\hat{n}\cdot\vec{\beta})R} \qquad \vec{A}(\vec{x},t) = \frac{q\vec{\beta}}{(1-\hat{n}\cdot\vec{\beta})R} \text{ (Evaluated at the retarded time)} \qquad t' = t - \frac{R}{c}$$

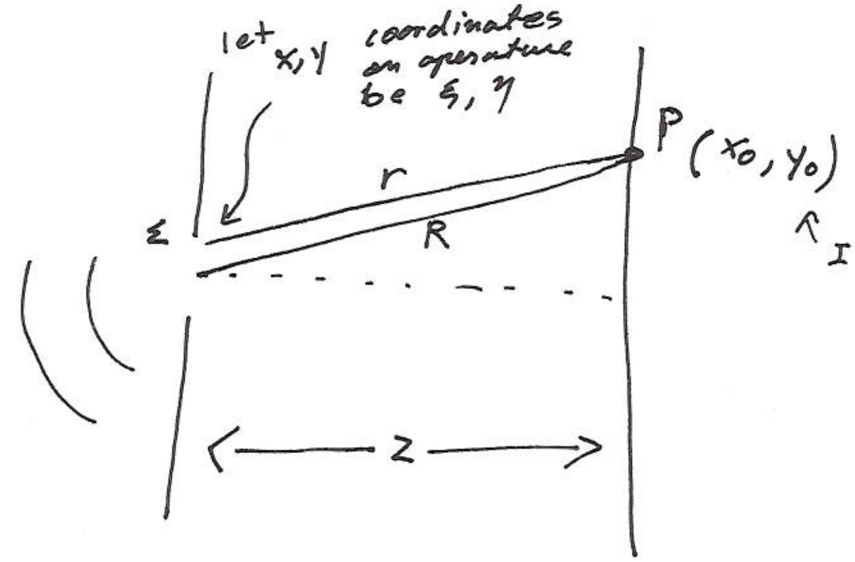
$$\psi(P) = \frac{1}{4\pi}\int_S \left[\frac{\partial\psi}{\partial n}\left(\frac{e^{ikr}}{r}\right) - \psi\,\frac{\partial}{\partial n}\left(\frac{e^{ikr}}{r}\right)\right] da$$

$$\psi(P) = \frac{1}{i\lambda}\int_{S_1} \psi(P_1)\frac{e^{ikr}}{r}\cos\theta da$$

$$\psi(u,v) = \frac{e^{ikz}}{i\lambda z} e^{i\frac{k}{2z}\left(u^2+v^2\right)} \int_{-\infty}^{\infty} \left[\psi(x,y) e^{i\frac{k}{2z}\left(x^2+y^2\right)} \right] e^{-i\frac{2\pi}{\lambda z}(ux+vy)} dxdy$$

$$\psi(u,v) = \frac{e^{ikz}}{i\lambda z} e^{i\frac{k}{2z}\left(u^2+v^2\right)} \int_{-\infty}^{\infty} \psi(x,y) e^{-i\frac{2\pi}{\lambda z}(ux+vy)} dxdy$$

3. a.



I choose general x_0, y_0 instead of $(0, Y)$ as the problem says.

The incoming spherical wave is

$$\psi_{inc} = A \frac{e^{-ikr}}{r}$$

$$r^2 = (x_0 - \eta)^2 + (y_0 - \xi)^2 + z^2$$

$$= z^2 \left(1 + \frac{(x_0-\eta)^2 + (y_0 - \xi)^2}{z^2}\right)^{1/2}$$

$$r \simeq z + \frac{(x_0 - \eta)^2 + (y_0 - \xi)^2}{2z}$$

so

$$\psi_{inc} \simeq \frac{A}{z} e^{-ikz} e^{-\frac{ik}{2z}\left[(x_0-\eta)^2 + (y_0 - \xi)^2\right]}$$

b. The Diffraction Integral takes the form

$$\psi(x,y) = \frac{K}{2\pi i} \int_{\Sigma} d\xi\, d\eta \; \frac{e^{ikr}}{r} \, \psi_{inc}(\xi,\eta)$$

(the obliquity factor ≃ 1 for small angles)

Expanding r as we did before:

$$r^2 = (x-\eta)^2 + (y-\xi)^2 + z^2$$

We have a choice as to what approximations to make, which are valid in different regions.

- Continuing as in part a, we can write

$$r \simeq z + \frac{(x-\eta)^2 + (y-\xi)^2}{2z}$$

"Fresnel Diffraction"

- or, instead expand the squares,

$$r^2 = \underbrace{x^2+y^2+z^2}_{R^2} - 2x\eta - 2y\xi + \eta^2 + \xi^2$$

$$= R^2 \left(1 - 2\frac{(x\eta + y\xi)}{R^2} + \frac{\eta^2+\xi^2}{R^2} \right)$$

approximately negligible if $x \gg \eta$ and $y \gg \xi$

so

$$r \simeq z - \frac{(x\eta + y\xi)}{z}$$

I have used $R \simeq z$

"Fraunhofer Diffraction"

Fresnel Diffraction is valid in the near zone $F \geq 1$

Fraunhoffer Diffraction is valid in the far zone $F \ll 1$

where F is the Fresnel number

$F = \frac{a^2}{z\lambda}$ and a is a characteristic size of the aperture.

– When we say "Fresnel Diffraction," we mean near-zone diffraction, or

$$\Psi_{Fresnel}(x,y) = \frac{K}{2\pi i} \int \Psi_{inc}(\xi,\eta) \frac{e^{ikz}}{z} e^{\frac{iK}{2z}\left[(x-\eta)^2+(y-\xi)^2\right]} d\xi\, d\eta$$

square terms

– When we say "Fraunhofer Diffraction," we mean far-zone,

$$\Psi_{Fraunhofer}(x,y) = \frac{K}{2\pi i} \int \Psi_{inc}(\xi,\eta) \frac{e^{ikz}}{z} e^{-\frac{iK}{z}(x\eta+y\xi)} d\eta\, d\xi$$

linear terms, Fourier Transform

So, for the spherical incoming wave,

$$\Psi_{Fresnel} = \frac{K}{2\pi i}\frac{e^{ikz}}{z}\frac{Ae^{-ikz}}{z}\int \underbrace{e^{-\frac{iK}{2z}\left[(x_0-\eta)^2+(y_0-\xi)^2\right]} e^{\frac{iK}{2z}\left[(x-\eta)^2+(y-\xi)^2\right]}}_{e^{\frac{iK}{2z}(x^2-x_0^2+y^2-y_0^2)}\, e^{-\frac{iK}{z}\left[(x-x_0)\eta+(y-y_0)\xi\right]}} d\eta\, d\xi$$

$$= \text{constants} \cdot \int_{\Sigma} d\eta\, d\xi\; e^{-\frac{iK}{z}\left[(x-x_0)\eta+(y-y_0)\xi\right]}$$

but we see that this is exactly the Fraunhofer Diffraction pattern of the aperture, centered at (x_0,y_0) ie, if a plane wave was incident on the aperture.

2. a. $\rho(x) = q\,\delta(x)\delta(y)\left[\delta(z-(R_0+\rho\sin\omega t)) - \delta(z+(R_0+\rho\sin\omega t))\right]$

Note: there are higher multipoles, but we are ignoring them.

the dipole moment for this configuration is

$$\vec{P} = \int \vec{x}\rho(\vec{x}) = \hat{z}\,q\int z\left(\delta(z-(R_0+\rho\sin\omega t)) - \delta(z+(R_0+\rho\sin\omega t))\right)$$

$$= q\left[R_0+\rho\sin\omega t - \left(-(R_0+\rho\sin\omega t)\right)\right]\hat{z}$$

$$= (2qR_0 + 2q\rho\sin\omega t)\,\vec{z}$$

no time dependence, does not radiate

radiating dipole $\vec{p} = 2q\rho\,\hat{z}$

$$\frac{dP}{d\Omega} = \frac{c^2k^4Z_0}{32\pi^2}\left|\hat{r}\times\hat{z}\right|^2 4q^2\rho^2$$

$$= \frac{\omega^4}{8\pi c^2} Z_0\, q^2\rho^2\sin^2\theta$$

$\hat{r}\times\hat{z} = \sin\theta\cos\phi\,\hat{y}$
$\sin\theta\sin\phi\,-\hat{x}$

$|\hat{r}\times\hat{z}|^2 = \sin^2\theta$

$$\int\sin^2\theta\,d\Omega = 2\pi\int\sin^3\theta\,d\theta$$

$\int\sin\theta(1-\cos^2\theta)$

$\int_0^\pi\sin\theta - \int_0^\pi\cos^2\theta\sin\theta$

$u = \cos\theta$
$du = -\sin\theta\,d\theta$

$-\cos\theta\Big|_0^\pi + \int_1^{-1}u^2\,du$

$\frac{u^3}{3}\Big|_1^{-1}$

$-[-1-1] + \frac{1}{3}[-1-1] = 2 - \frac{2}{3} = \frac{4}{3}$

$= \frac{8\pi}{3}$

so, $\boxed{P = \frac{q^2\rho^2\omega^4 Z_0}{3\pi c^2}}$

b, for 1 charge,

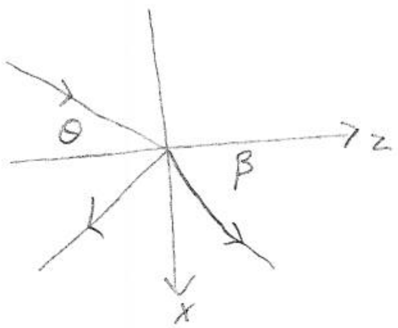
$\vec{P} \Rightarrow qR_0 + q\rho \sin\omega t$ so the dipole moment is halved

Since the Power radiated $\sim p^2$, $\frac{1}{4}$ the power is radiated

for a uniform charge

$\vec{P} \to 0$ we have a monopole -> no radiation.

3. a. The transmitted wave is $\vec{E}_T = E_T \hat{y}\, e^{i(\vec{K}_T \cdot \vec{x})}$



$$\vec{H}_T = \frac{1}{\eta_2} \hat{K}_T \times \vec{E}_T$$

$$\vec{K}_T = \frac{\omega}{c} \begin{bmatrix} \sin\beta \\ 0 \\ \cos\beta \end{bmatrix}$$

Snells Law: $n \sin\theta = \sin\beta \Rightarrow \sin\theta_c = \frac{1}{n}$

$$\cos\beta = \sqrt{1-\sin^2\beta} = \sqrt{1-n^2\sin^2\theta}$$

$$= \left[1 - \left(\frac{\sin\theta}{\sin\theta_c}\right)^2\right]^{1/2}$$

- E_T will be just a magnitude and a phase shift, it does not affect the propagation.
- Look at $e^{i(\vec{K}_T \cdot \vec{x})}$

$$= e^{i\frac{\omega}{c}\left[\underset{= n\sin\theta}{\sin\beta}\, x + \cos\beta\, z\right]}$$

$$\cos\beta = \underbrace{\left[1 - \frac{\sin\theta^2}{\sin\theta_c^2}\right]^{1/2}}_{\text{which is imaginary for } \theta > \theta_c}$$

$$= i\left[\left(\frac{\sin\theta}{\sin\theta_c}\right)^2 - 1\right]^{1/2}$$

$$= \underbrace{e^{in\frac{\omega}{c}\sin\theta\, x}}_{\text{Propagation in } x} \; \underbrace{e^{-\frac{\omega}{c}\sqrt{\frac{\sin\theta^2}{\sin\theta_c^2}-1}\; z}}_{\text{decay in } z}$$

δ is defined by

$$e^{-\frac{\omega}{c}\sqrt{\frac{\sin^2}{\sin^2}-1}\,\delta} = e^{-1}$$

so $$\delta = \frac{c}{\omega\sqrt{\frac{\sin^2\theta}{\sin^2\theta_c}-1}} = \boxed{\frac{\lambda}{2\pi\left[\frac{\sin^2\theta}{\sin^2\theta_c}-1\right]^{1/2}}}$$

b. The energy Transport across the boundary is

$$\hat{z} \cdot \langle \vec{S} \rangle = \frac{1}{2} \hat{z} \cdot Re\left(\vec{E} \times \vec{H}^*\right) = \frac{|E_T|^2}{2\eta} Re\left[\hat{z} \cdot \left(\hat{y} \times \hat{k}_T \times \hat{y}\right)\right]$$

$$\hat{z} \cdot \begin{bmatrix} \sin\beta \\ 0 \\ \cos\beta \end{bmatrix} = \cos\beta$$

but $\cos\beta$ is purely imaginary

so $Re\left[\cos\beta\right] = 0$

$\Rightarrow \quad \hat{z} \cdot \langle \vec{S} \rangle = 0$

4. Lab

a. $p_1^\mu = \begin{bmatrix} E/c \\ \vec{p}_1 \end{bmatrix}$ $\quad p_2^\mu = \begin{bmatrix} mc \\ \vec{0} \end{bmatrix}$ $\quad p^\mu = p_1^\mu + p_2^\mu$

CoM

$p_1'^\mu = \begin{bmatrix} \frac{W}{2c} \\ p' \end{bmatrix}$ $\quad p_2'^\mu = \begin{bmatrix} \frac{W}{2c} \\ -p' \end{bmatrix}$ $\quad p'^\mu = p_1'^\mu + p_2'^\mu$

where W is the total energy in the CoM frame

From Lorentz invariance

$$p^\mu = \begin{bmatrix} \frac{E+mc^2}{c} \\ \vec{p}_1 \end{bmatrix} \quad \text{and} \quad p'^\mu = \begin{bmatrix} \frac{W}{c} \\ \vec{0} \end{bmatrix}$$

are related by

$$x_0 = \gamma_{com}(x_0' + \beta_{com} x_1') \quad \Rightarrow \quad E + mc^2 = \gamma W$$

$$x_1 = \gamma_{com}(x_1' + \beta x_0') \quad \Rightarrow \quad p_1 = \gamma \beta \frac{W}{c}$$

$$\text{so} \quad \beta = \frac{p_1 c}{\gamma W} = \frac{p_1 c}{E + mc^2}$$

now, p_1 is related to E, $\left(\frac{E}{c}\right)^2 - p_1^2 = m^2c^2$

$$p_1^2 = \left(\frac{E}{c}\right)^2 - m^2c^2$$

$$pc = \sqrt{E^2 - m^2c^4}$$

so, $$\boxed{\beta_{com} = \frac{\sqrt{E^2 - m^2c^4}}{E + mc^2}}$$

b. $p^\mu p_\mu = p'^\mu p'_\mu$

$$(p_1^\mu + p_2^\mu)(p_{1\mu} + p_{2\mu}) = p_1^\mu p_{1\mu} + p_2^\mu p_{2\mu} + 2 p_1^\mu p_{2\mu} = 2m^2c^2 + 2EM$$

$$p'^\mu p'_\mu = \left(\frac{W}{c}\right)^2$$

so, $\boxed{W^2 = 2mc^2(mc^2 + E)}$

c. After the collision, in the CoM frame, $p_1'^\mu = \begin{bmatrix} \frac{W}{2c} \\ -p' \end{bmatrix}$ simply changes direction

$p_2'^\mu = \begin{bmatrix} W/2c \\ +p' \end{bmatrix}$

expressing p' in terms of the initial momentum p_1,

$$p' = \gamma\left(p_1 - \beta \frac{E}{c}\right)$$

transforming the final momentum in the CoM from back into the lab frame,

$$p_{1\text{final}} = \gamma\left(-p' + \beta \frac{W}{2c}\right)$$ and plug in p' from above

$$= \gamma\left(-\gamma\left(p_1 - \beta\frac{E}{c}\right) + \beta \frac{W}{2c}\right) = -\gamma^2\left(p_1 - \beta\frac{E}{c}\right) + \frac{\beta}{2c}\gamma W$$

γW: $E + mc^2$ from pt a

β: $\frac{\sqrt{E^2 - m^2c^4}}{E + mc^2}$ from pt a

$$\gamma^2 = \frac{1}{1-\beta^2} = \frac{E + mc^2}{2mc^2}$$

$$= -\frac{E + mc^2}{2mc^2}\left(p_1 - \frac{E}{c}\frac{\sqrt{E^2 - m^2c^4}}{E + mc^2}\right) + \frac{1}{2c}\sqrt{E^2 - m^2c^4}$$

p_1: $\frac{1}{c}\sqrt{E^2 - m^2c^4}$

$$= -\frac{1}{c}\frac{\sqrt{E^2 - m^2c^4}}{2mc^2}\left(E + mc^2 - E\right) + \frac{1}{2c}\sqrt{E^2 - m^2c^4}$$

$$= -\frac{1}{2c}\sqrt{E^2 - m^2c^4} + \frac{1}{2c}\sqrt{E^2 - m^2c^4} = 0$$

So the first particle stops,

the second particle...

$$P_{2final} = \gamma\left(p' + p\frac{w}{2c}\right)$$

following the same procedure, the last 2 terms add instead of subtract.

$$= \frac{1}{c}\sqrt{E^2 - m^2c^4}$$

$$\gamma m v = \frac{1}{c}\sqrt{E^2 - m^2c^4} \quad \Rightarrow \quad \frac{c^2m^2v^2}{\left(1-\left(\frac{v}{c}\right)^2\right)} = E^2 - m^2c^4$$

$$v^2(c^2m^2) = (E^2 - m^2c^4)\left(1-\left(\frac{v}{c}\right)^2\right)$$

$$v^2(c^2m^2) = -\left(\left(\frac{E}{c}\right)^2 - m^2c^2\right)v^2 + (E^2 - m^2c^4)$$

$$v^2\left(c^2m^2 + \left(\frac{E}{c}\right)^2 - m^2c^2\right) = E^2 - m^2c^4$$

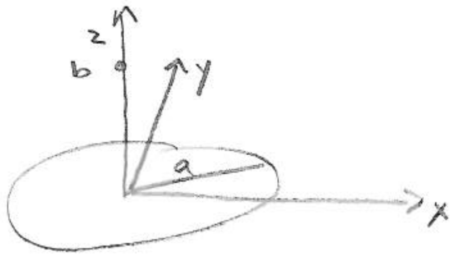
$$\frac{v^2}{c^2} = \frac{E^2 - m^2c^4}{E^2}$$

$$= 1 - \frac{m^2c^4}{\gamma_i^2 m^2c^4} = 1 - \frac{1}{\gamma_i^2} = \beta_i^2$$

So $\beta_f^2 = \beta_i^2$

Thus if the particles move in the same direction after the collision as before the collision, the moving particle stops and the other particle gains the same velocity as the incoming particle had.

5.



a. $t' = t - \frac{R}{c} = t - \frac{1}{c}[b^2+a^2]^{1/2}$

b. $\rho = q\,\delta(z)\,\delta(x - a\cos\omega t)\,\delta(y - a\sin\omega t)$

$$\phi(\vec{x},t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(x', t_{ret})}{\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{1/2}}\,d^3x'$$

$$\phi(0,0,b,t) = \frac{q}{4\pi\varepsilon_0}\int \frac{\delta(z')\,\delta(x'-a\cos\omega t')\,\delta(y-a\sin\omega t')}{\left[x'^2+y'^2+(b-z')^2\right]^{1/2}}$$

$$= \frac{q}{4\pi\varepsilon_0\,[a^2+b^2]^{1/2}}$$

c. $\vec{J} = q\,\delta(\vec{x}-\vec{x}')\,\vec{v}$

$\vec{x}' = a(\cos\omega t\,\hat{x} + \sin\omega t\,\hat{y})$

$\dot{\vec{x}}' = a\omega(-\sin\omega t\,\hat{x} + \cos\omega t\,\hat{y})$

$$= q\,\delta(z)\,\delta(x-a\cos\omega t)\,\delta(y-a\sin\omega t)\,a\omega\,(-\sin\omega t\,\hat{x} + \cos\omega t\,\hat{y})$$

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}_{ret}}{\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{1/2}} = \frac{\mu_0 a\omega q}{4\pi}\int \frac{\delta(z')\,\delta(x'-a\cos\omega t')\,\delta(y'-a\sin\omega t')(-\sin\omega t'\hat{x}+\cos\omega t'\hat{y})}{[\quad]^{1/2}}$$

$$\vec{A}(0,0,b,t) = \frac{\mu_0 a\omega q}{4\pi\,[a^2+b^2]^{1/2}}\left[-\sin\omega t'\,\hat{x} + \cos\omega t'\,\hat{y}\right]$$

$$= \frac{\mu_0 a\omega q}{4\pi\,(a^2+b^2)^{1/2}}\,[\hat{y} - i\hat{x}]\,e^{-i\omega t'}$$

PHYSICS 210B, Winter 2010
Midterm Exam (90 points in total)

- Closed books and closed notes
- Please write down the necessary intermediate steps
- Write your answers in the space provided. Continue on the back if necessary.
- A formula sheet is provided at the end of the exam.

Name:____________________________ ID: ______________________________.

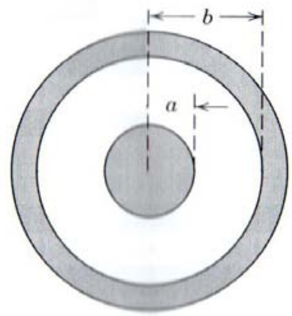
Problem 1: _________________________

Problem 2: _________________________

Problem 3: _________________________

Total: _________________________

1. A transmission line consisting of two concentric circular cylinders of metal with conductivity σ and skin depth δ, as shown, is filled with a uniform lossless dielectric (μ, ε). A TEM mode is propagated along this line.



(a) Show that the time-averaged power flow along the line is $P = \sqrt{\frac{\mu}{\varepsilon}} \pi a^2 |H_0|^2 \ln\left(\frac{b}{a}\right)$, where H_0 is the peak value of the azimuthal magnetic field at the surface of the inner conductor. (10 points).

(b) Show that the transmitted power is attenuated along the line as $P(z) = Pe^{-2\gamma z}$ where $\gamma = \frac{1}{2\sigma\delta}\sqrt{\frac{\varepsilon}{\mu}}\frac{(1/a + 1/b)}{\ln(b/a)}$. (10 points)

(c) The characteristic impedance Z_0 of the line is defined as the ratio of the voltage between the cylinders to the axial current flowing in one of them at any position z. Show that for this line $Z_0 = \frac{1}{2\pi}\sqrt{\frac{\mu}{\varepsilon}} \ln(b/a)$. (10 points).

2. A pulsar emits bursts of radio waves which are observed from the Earth at two different frequencies, ω_1 and ω_2. An astronomer notes that the arrival time of the bursts is delayed at the lower frequency, that is, the pulse at ω_1 arrives after the pulse at ω_2. The delay, τ is due to dispersion in the interstellar medium. Assume this medium consists of ionized hydrogen (called the dilute plasma).
   (a) Find the index of refraction of the dilute plasma. (To get full credit, you need to first write down the equation of the motion of a free electron in an oscillating electric wave). (15 points)
   (b) Find the distance from the pulsar to the Earth. (15 points)

   [Assume m is the mass of the electron and N the number of electrons per unit volume.]

3. Consider a "classical" hydrogen atom with the electron moving in a circular orbit at the Bohr radius $a_0 = \frac{4\pi\varepsilon_0\hbar^2}{me^2}$. Assume the electron's orbit is in the x-y plane with the period T.
   a. Find the lowest multipole moment. (10 points)
   b. Calculate the electric and magnetic fields in the radiation zone. (10 points)
   c. Calculate the power radiated per unit solid angle. (10 points)

   [Note, you just need to calculate (b) and (c) for the lowest multipole moment].

Formula Sheet for Midterm

You may use any of the following equations without derivation.

$$\vec{P}=\varepsilon_0\chi_e\vec{E} \qquad \varepsilon=(1+\chi_e)\varepsilon_0$$

$$\vec{E}=\vec{E}_0e^{-i(\omega t-\vec{k}\cdot\vec{r})} \qquad \vec{B}=\vec{B}_0e^{-i(\omega t-\vec{k}\cdot\vec{r})} \qquad \vec{B}=\sqrt{\mu\varepsilon}\hat{k}\times\vec{E} \qquad v=\frac{\omega}{k}=\frac{1}{\sqrt{\varepsilon\mu}}$$

$$n=\frac{c}{v} \qquad k=\frac{n\omega}{c} \qquad \vec{\nabla}\cdot\vec{E}=\frac{\rho}{\varepsilon_0}$$

$$D_1^\perp-D_2^\perp=\sigma_f \qquad B_1^\perp=B_2^\perp \qquad E_1^\parallel=E_2^\parallel \qquad \vec{H}_1^\parallel-\vec{H}_2^\parallel=\vec{K}_f\times\hat{n}$$

$$n\sin\theta=n'\sin\theta' \qquad <\vec{S}'>=\frac{1}{2}\mathrm{Re}\left(\vec{E}'\times\vec{H}'^*\right)$$

$$\gamma=-\frac{1}{2P}\frac{dP}{dz} \qquad \frac{dP}{dz}=-\frac{1}{2\sigma\delta}\oint_C\left|\hat{n}\times\vec{H}\right|^2dl$$

$$k^2=\mu\varepsilon\omega^2+i\mu\sigma\omega \qquad k=k_1+ik_2=\omega\sqrt{\mu\varepsilon\left(1+\frac{i\sigma}{\varepsilon\omega}\right)} \qquad \delta=1/k_2 \qquad \alpha=2k_2$$

TE waves $\qquad \vec{\nabla}_tB_z=-\frac{ik_c^2}{k_g}\vec{B}_t \qquad \vec{B}_t=\frac{k_g}{\omega}\left(\hat{z}\times\vec{E}_t\right)$

TM waves $\qquad \vec{\nabla}_tE_z=-\frac{ik_c^2}{k_g}\vec{E}_t \qquad \vec{E}_t=-\frac{k_g}{\mu\varepsilon\omega}\left(\hat{z}\times\vec{B}_t\right)$

$$k_0^2=k_c^2+k_g^2$$

$$\vec{p}=\int\vec{x}'\rho(\vec{x}')d^3x' \qquad \vec{A}_{ED}(\vec{x})=-\frac{i\mu_0\omega\vec{p}}{4\pi}\frac{e^{ikr}}{r} \qquad \vec{B}_{ED}=\frac{\mu_0ck^2}{4\pi}\frac{e^{ikr}}{r}\hat{n}\times\vec{p}$$

$$\vec{E}_{ED}=-\frac{k^2}{4\pi\varepsilon_0}\frac{e^{ikr}}{r}\left[\hat{n}\times(\hat{n}\times\vec{p})\right]=Z_0\vec{H}_{ED}\times\hat{n} \qquad Z_0=\sqrt{\frac{\mu_0}{\varepsilon_0}} \qquad \frac{dP}{d\Omega}=\frac{c^2k^4Z_0}{32\pi^2}\left|\hat{n}\times\vec{p}\right|^2$$

$$\vec{m}=\frac{1}{2}\int(\vec{x}'\times\vec{J})d^3x' \qquad \vec{A}_{MD}(\vec{x})=\frac{\mu_0ik}{4\pi}\frac{e^{ikr}}{r}\hat{n}\times\vec{m} \qquad \vec{B}_{MD}=\frac{\mu_0k^2}{4\pi}\frac{e^{ikr}}{r}(\hat{n}\times\vec{m})\times\hat{n}$$

$$\vec{E}_{MD}=\frac{Z_0k^2}{4\pi}\frac{e^{ikr}}{r}\hat{n}\times\vec{m} \qquad \frac{dP}{d\Omega}=\frac{k^4Z_0}{32\pi^2}\left|\hat{n}\times\vec{m}\right|^2$$

$$\vec{A}_{EQ}(\vec{x})=-\frac{\mu_0ck^2}{8\pi}\frac{e^{ikr}}{r}\int\vec{x}'(\hat{n}\cdot\vec{x}')\rho(\vec{x}')d^3x' \qquad \vec{B}_{EQ}=-\frac{\mu_0ick^3}{24\pi}\frac{e^{ikr}}{r}\hat{n}\times\vec{Q}(\hat{n})$$

$$\vec{E}_{EQ}=Z_0(\vec{H}_{EQ}\times\hat{n}) \qquad \frac{dP}{d\Omega}=\frac{c^2k^6Z_0}{1152\pi^2}\left|\hat{n}\times\vec{Q}(\hat{n})\right|^2 \qquad Q_i(\hat{n})=\sum_jQ_{ij}n_j$$

Midterm Solutions

1. Since we have a TEM mode, $\vec{E}_t$ is a solution to an electrostatics problem

$$\vec{E} = \frac{E_0}{\rho}\hat{\rho} \qquad \vec{H} = \frac{1}{\eta}\hat{z}\times\vec{E} = \frac{1}{\eta}E_0\frac{1}{\rho}\hat{\phi} = \frac{H_0 a}{\rho}\hat{\phi}$$

in terms of H_0,
$$\boxed{\begin{aligned}\vec{E}_t &= \eta H_0 \frac{a}{\rho}\hat{\rho}\\ \vec{H}_t &= H_0\frac{a}{r}\hat{\phi}\end{aligned}}$$

a. $$\langle S\rangle = \frac{1}{2}\vec{E}\times\vec{H}^* = \frac{1}{2}\eta|H_0|^2\frac{a^2}{r^2}\hat{z}$$

$$\langle \hat{P}\rangle = \int \langle S\rangle\cdot z\,da = \frac{1}{2}\eta|H_0|^2a^2\int\frac{1}{r^2}\,r\,dr\,d\phi = \boxed{\pi a^2\eta|H_0|^2\ln b/a}$$

b. $$-\frac{dP}{da} = \frac{1}{2\sigma\delta}\left|\hat{n}\times\vec{H}_{\parallel}\right|^2$$

$$-\frac{\partial P}{\partial z} = -\int_{\text{conductors}}\frac{\partial P}{\partial a}\,dl = -\frac{1}{2\sigma\delta}\left[\int_{b\text{ surface}}|H|^2d\phi + \int_{a\text{ surface}}|H|^2d\phi\right]$$

$$= -\frac{1}{2\sigma\delta}\left[2\pi b\left(H_0\frac{a}{b}\right)^2 + 2\pi a\left(H_0\right)^2\right]$$

$$= -\frac{\pi a^2}{\sigma\delta}\left[\frac{1}{b}+\frac{1}{a}\right]|H_0|^2$$

$$\gamma = -\frac{1}{2P}\frac{\partial P}{\partial z} = -\frac{1}{2}\,\frac{-\frac{\pi a^2}{\sigma\delta}\left[\frac{1}{b}+\frac{1}{a}\right]|H_0|^2}{\pi a^2\eta|H_0|^2\ln b/a} = \boxed{\frac{1}{2\sigma\delta}\sqrt{\frac{\varepsilon}{\mu}}\;\frac{\left[\frac{1}{b}+\frac{1}{a}\right]}{\ln b/a}}$$

c. $$Z = \frac{V}{I} \qquad V = \int_a^b \vec{E}\cdot dr = \eta H_0 a\ln b/a$$

From Ampere's law

$$\oint H\,d\phi = I$$

$$2\pi r\left(H_0\frac{a}{r}\right) = I$$

$$Z = \frac{\eta H_0 a\ln b/a}{2\pi H_0 a} = \boxed{\frac{1}{2\pi}\sqrt{\frac{\mu}{\varepsilon}}\ln b/a}$$

2. a. Write the equation of motion for a single electron:

$$M\ddot{x} = qE - Kx$$ ← same binding constant $\omega_0^2 = \frac{K}{M}$

$$\vec{E} = E_0 \hat{x} e^{i\omega t}$$

$$\ddot{x} = \frac{q}{M} E - \omega_0^2 x \qquad x = x_0 e^{i\omega t} \qquad \ddot{x} = -\omega^2 x$$

$$(\omega_0^2 - \omega^2) x_0 = \frac{q}{M} E_0$$

$$\Rightarrow \quad x_0 = \frac{\frac{q}{M} E_0}{(\omega_0^2 - \omega^2)}$$ is the Amplitude of oscillation

Since the electron is "free" ω_0 is actually zero, we could ignore it here if we want. Also $\gamma = 0$ since there is no damping on a "free" electron.

The dipole moment of this electron is $\vec{p} = q x_0 \hat{x} e^{i\omega t}$

$$|p| = \frac{\frac{q^2}{M} E_0}{\omega_0^2 - \omega^2}$$

recall: $\vec{D} = \varepsilon \vec{E} = \varepsilon_0 (1 + \chi) \vec{E}$ and $\vec{P} = \varepsilon_0 \chi \vec{E}$ ← Polarization, Dipole moment/Volume

if 1 electron contributes p,

then $\vec{P} = N\vec{p}$ where $N = \frac{\#\text{ electrons}}{\text{Volume}}$

$$|P| = \frac{\frac{Nq^2}{M} E_0}{\omega_0^2 - \omega^2} = \varepsilon_0 \chi E_0$$

$$\text{so,} \quad \chi = \frac{Nq^2}{M\varepsilon_0} \frac{1}{\omega_0^2 - \omega^2}$$

for $\omega^2 \gg \omega_0^2$ plasma limit (ω_0 is zero after all)

$$\chi = -\frac{\omega_p^2}{\omega^2} \qquad \omega_p = \sqrt{\frac{Nq^2}{m\varepsilon_0}}$$

thus

$$\frac{\varepsilon}{\varepsilon_0} = 1 - \frac{\omega_p^2}{\omega^2}$$

and

$$\boxed{n = \sqrt{\frac{\varepsilon}{\varepsilon_0}} = \left[1 - \frac{\omega_p^2}{\omega^2}\right]^{1/2} \qquad \omega_p = \sqrt{\frac{Nq^2}{m\varepsilon_0}}}$$

b. The pulse the pulsar emitts is a wavepacket that travels to earth. As such, it travels at the group velocity, not phase velocity.

$$n = \left[1 - \frac{\omega_p^2}{\omega^2}\right]^{1/2} \quad \text{from pt a,} \quad K = \frac{\omega}{c} n = \frac{\omega}{c}\left[1 - \frac{\omega_p^2}{\omega^2}\right]^{1/2}$$

$$\frac{1}{V_g} = \frac{\partial K}{\partial \omega} = \frac{1}{c}\left[\frac{\omega}{2}\left[1 - \frac{\omega_p^2}{\omega^2}\right]^{-1/2}\left(\frac{2\omega_p^2}{\omega^3}\right) + \left[1 - \frac{\omega_p^2}{\omega^2}\right]^{1/2}\right]$$

$$= \frac{1}{c}\left[\frac{\omega_p^2}{\omega^2}\left[1 - \frac{\omega_p^2}{\omega^2}\right]^{-1/2} + \frac{\left[1 - \frac{\omega_p^2}{\omega^2}\right]}{\left[1 - \omega_p^2/\omega^2\right]^{1/2}}\right]$$

$$= \frac{1}{c}\left[\frac{\frac{\omega_p^2}{\omega^2} + 1 - \frac{\omega_p^2}{\omega^2}}{\left[1 - \frac{\omega_p^2}{\omega^2}\right]^{1/2}}\right] = \frac{1}{c}\left[1 - \frac{\omega_p^2}{\omega^2}\right]^{-1/2}$$

$$\text{so } V_g = c\left[1 - \frac{\omega_p^2}{\omega^2}\right]^{1/2}$$

$$t_1 = \frac{d}{V_1} \qquad t_2 = \frac{d}{V_2}$$

$$\tau = t_1 - t_2 = d\left[\frac{1}{V_g^1} - \frac{1}{V_g^2}\right] = \frac{d}{c}\left[\left[1 - \frac{\omega_p^2}{\omega_1^2}\right]^{-1/2} - \left[1 - \frac{\omega_p^2}{\omega_2^2}\right]^{-1/2}\right]$$

if $\omega >> \omega_p$,

$$\approx \frac{d}{c}\left[1 + \frac{\omega_p^2}{2\omega_1^2} - 1 - \frac{\omega_p^2}{2\omega_2^2}\right] = \boxed{\frac{d\,\omega_p^2}{c \cdot 2}\left(\frac{1}{\omega_1^2} - \frac{1}{\omega_2^2}\right)}$$

Notes on phase velocity and group velocity (what I was thinking when convincing myself that we need to use v_g)

the pulse emitted by the pulsar is composed of several frequencies

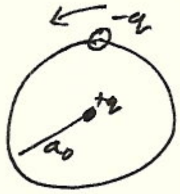
imagine  built up from A Fourier transform in K (space).

- If there is no dispersion, the pulse travels at speed c, which, by the way, is still the group velocity since $\omega = cK$, but also happens to be the phase velocity of every frequency.

- Remember that the Fourier components of this pulse theoretically extend to infinity, and simply interfere destructively with other components outside the pulse to give a sum of zero. When you calculate the phase velocity, you calculate the velocity of a point on one component wave. This point can travel outside the pulse and does not reflect something we can measure. Notice the phase velocity for a wave traveling through a plasma is greater than c; $n = \left[1 - \frac{\omega_p^2}{\omega^2}\right]^{1/2}$ is always less than 1.

- t_1 and t_2 are the times for a packet centered at ω_1 or ω_2 to reach earth. Each one travels at v_g.

3. a.



$T = \frac{2\pi}{\omega}$

This configuration has a dipole term as the lowest order.

By inspection, $\vec{p} = -q a_0 [\hat{x} + i\hat{y}]$

two dipoles oscillating out of phase and $\perp$ to each other.

If you don't "see" $\vec{p} = -q a_0 [\hat{x} + i\hat{y}]$

here is the long way: $\rho(\vec{x}) = q\left[\delta(\vec{r}) - \frac{1}{r^2 \sin\theta}\delta(r-a_0)\,\delta(\cos\theta)\,\delta(\phi-\omega t)\right]$

write $\delta(\phi-\omega t)$ as a Fourier series in time

sum over $\pm m$

$$\delta(\phi-\omega t) = \sum_m A_m e^{-im\omega t}$$

multiply by $e^{in\omega t}$ and integrate $\int_0^T dt$

$$\int_0^T dt\, \underbrace{\delta(\phi-\omega t)}_{=\frac{1}{\omega}\delta\left(t-\frac{\phi}{\omega}\right)} e^{in\omega t} = \sum_m A_m \underbrace{\int_0^T e^{i(n-m)\omega t}\, dt}_{\frac{2\pi}{\omega}\delta_{n,m}} \Rightarrow A_m = \frac{e^{im\phi}}{2\pi}$$

$$\rho(x,t) = q\,\delta(\vec{r}) - \frac{q\,\delta(r-a_0)\,\delta(\cos\theta)}{2\pi\, r^2 \sin\theta}\sum_m e^{im\phi} e^{-im\omega t}$$

we see that the lowest radiating term is $m = 1$,

$$\rho_1 = \frac{-q}{r^2\sin\theta}\,\frac{\delta(r-a_0)\,\delta(\cos\theta)}{2\pi}\left[e^{i(\phi-\omega t)} + e^{-i(\phi-\omega t)}\right] = \frac{-q}{2\pi r^2 \sin\theta}\,\delta(r-a_0)\,\delta(\cos\theta)\; 2\cos(\phi-\omega t)$$

for $m = \pm 1$

find the dipole moment:

$$\vec{p} = \int \vec{x}\,\rho_1\, d^3x = \frac{-2q}{2\pi}\iint \begin{bmatrix} r\sin\theta\cos\phi \\ r\sin\theta\sin\phi \\ r\cos\theta \end{bmatrix} \frac{\delta(r'-a_0)\,\delta(\cos\theta')}{r'^2 \sin\theta'} \cos(\phi-\omega t)\, r'^2 dr\, \sin\theta' d\theta\, d\phi$$

integrate r, θ

$$= \frac{-q}{\pi} a_0 \int d\phi \begin{bmatrix} \cos\phi \\ \sin\phi \\ 0 \end{bmatrix} \cos(\phi-\omega t)$$

$\cos\phi\cos\omega t + \sin\phi\sin\omega t$

use $\int_0^{2\pi} d\phi \sin^2\phi = \int_0^{2\pi} d\phi\cos^2\phi = \pi$

$\int_0^{2\pi} d\phi \sin\phi\cos\phi = 0$

$$= -\frac{q a_0}{\pi}\cdot\pi \begin{bmatrix} \cos\omega t \\ \sin\omega t \end{bmatrix} = -q a_0 [\cos\omega t\,\hat{x} + \sin\omega t\,\hat{y}]$$

$$= -q a_0\, \mathrm{Re}\left[(\hat{x} + i\hat{y})\, e^{-i\omega t}\right]$$

verifying $\vec{p}$ above.

b. The rest is just plugging in.

$$\vec{B} = \frac{\mu_0 c k^2}{4\pi} \frac{e^{ikr}}{r} \left[\hat{r} \times \vec{p} \right] \qquad \vec{E} = -\frac{k^2}{4\pi\varepsilon_0} \frac{e^{ikr}}{r} \left[\hat{r} \times (\hat{r} \times \vec{p}) \right]$$

$$\hat{x} = \hat{r} \sin\theta\cos\phi + \hat{\theta}\cos\theta\cos\phi - \hat{\phi}\sin\phi$$
$$\hat{y} = \hat{r}\sin\theta\sin\phi + \hat{\theta}\cos\theta\sin\phi + \hat{\phi}\cos\phi$$
$$\hat{r} \times \hat{x} = \hat{\phi}\cos\theta\cos\phi + \hat{\theta}\sin\phi$$
$$\hat{r} \times \hat{y} = \hat{\phi}\cos\theta\sin\phi - \hat{\theta}\cos\theta$$

$$\hat{r} \times \left(\hat{x} + i\hat{y} \right)$$
$$= \hat{\phi}\cos\theta \left(\cos\phi + i\sin\phi \right)$$
$$\hat{\theta} \left(\sin\phi - i\cos\phi \right)$$
$$= \hat{\phi}\cos\theta\, e^{i\phi} - i\hat{\theta}\, e^{i\phi}$$

$$\boxed{\vec{B} = \frac{-\mu_0 c k^2 q a_0}{4\pi} \frac{e^{ikr}}{r} e^{i\phi} \left[\hat{\phi}\cos\theta - i\hat{\theta} \right]}$$

$$\hat{r} \times (\hat{r} \times \vec{p}) = -qa \left[\cos\theta\, e^{i\phi} (-\hat{\theta}) - i e^{i\phi} \hat{\phi} \right]$$

$$\boxed{\vec{E} = +\frac{q a_0 k^2}{4\pi\varepsilon_0} \frac{e^{ikr}}{r} e^{i\phi} \left[-\cos\theta\, \hat{\theta} - i\hat{\phi} \right]}$$

c.

$$\frac{\partial P}{\partial \Omega} = \frac{c^2 k^4 \eta}{32\pi^2} \left| \hat{n} \times \vec{p} \right|^2$$

$$= \boxed{\frac{q^2 a_0^2 c^2 k^4 \eta}{32\pi^2} \left[\cos^2\theta + 1 \right]}$$

PHYSICS 210B, Winter 2011
Final Exam (100 points in total)

- Closed books and closed notes
- Please write down the necessary intermediate steps
- Write your answers in the space provided. Continue on the back if necessary.
- A formula sheet is provided at the end of the exam.

Name:____________________________ ID: ______________________________.

Problem 1: _________________________

Problem 2: _________________________

Problem 3: _________________________

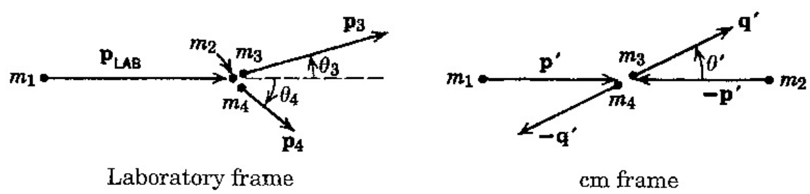
Problem 4: _________________________

Problem 5: _________________________

Total: _________________________

1. Consider a plane electromagnetic wave of wavelength λ is incident on a rectangular aperture. The width of the aperture is a and b along the X and Y-axes, respectively. Under the Fraunhofer approximation $\frac{ab}{\lambda x} << 1$ where x is the distance between the aperture and the detector plane, the diffraction intensity is related to the square of the Fourier transform of the aperture.
   (a) Calculated the Fraunhofer diffraction pattern of the aperture. (8 points)
   (b) Determine the maximum and minimum positions of the diffraction intensity. (6 points)
   (c) Verify the Heisenberg Uncertainty Principle based on this system. (6 points)

2. In a collision process a particle of mass m_2, at rest in the laboratory, is struck by a particle of mass m_1, momentum $\vec{p}_{lab}$ and total energy E_{lab}. In the collision the two initial particles are transformed into two others of mass m_3 and m_4. The configurations of the momentum vectors in the center-of-mass (CoM) frame and the laboratory frame are shown below.
   (a) Show that the total energy W in the CoM frame has its square given by $W^2 = m_1^2 + m_2^2 + 2m_2 E_{lab}$. (10 points)
   (b) Show that the 3-momentum in the CoM frame is $\vec{p}' = \frac{m_2 \vec{p}_{lab}}{W}$. (10 points)



3. Consider a charge q and mass m that is harmonically bound (frequency ω_0) along x (i.e. the charge is constrained to move on the x-axis). A plane wave propagating along z, $\vec{E} = E_0 e^{i(kz-\omega t)}\hat{x}$, is incident on the charge. Calculate the differential scattering cross-section $\frac{d\sigma}{d\Omega}$ as a function of the scattering angle θ. (20 points)

4. A thin linear antenna of length *d*, centered at the origin, and parallel to the *z* axis, is excited in such a way that the current (*I*) makes a full wavelength of sinusoidal oscillation at frequency ω.
   (a) Find the current density, $\vec{J}(\vec{x},t)$. (5 points)
   (b) Find the vector potential of the radiation field, $\vec{A}(\vec{x},t)$, in the far zone. (8 point)
   (c) Calculate the power radiated per unit solid angle, $\frac{dP}{d\Omega}$, in the far zone. (7 point)

   [Hint: if d, λ << r, then $\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int \vec{J}(\vec{x}')e^{-ik\hat{n}\cdot\vec{x}'}d^3x'$. $\vec{\nabla}\times\vec{A} = ik\hat{n}\times\vec{A}$]

5. A low-energy electron has a velocity $v_0 << c$ at infinity. The velocity $\vec{v}_0$ is directed towards a fixed, repulsive Coulomb field, the potential energy for which is given by $U(r) = \frac{Ze^2}{r}$. The electron is decelerated until it comes to rest and then is accelerated again in a direction opposite to the original direction of motion. Show that when the electron has again reached an infinite distance from the Coulomb scattering center, the kinetic energy of the electron is about $\frac{1}{2}mv^2 = \frac{1}{2}mv_0^2\left(1 - \frac{16v_0^3}{45Zc^3}\right)$ where m is the electron mass and the term depending on v_0^5 represents the energy radiated away during the deceleration and acceleration processes. [Hint: assume that the radiation reaction does not affect the dynamics appreciably. Also, if you can write down all the main steps, you will get most of the points. The detailed calculation is less important.] (20 points)

Formula Sheet for the Final Exam

You may use any of the following equations without derivation.

$$\vec{E} = \vec{E}_0 e^{-i(\omega t - \vec{k}\cdot\vec{r})} \qquad \vec{B} = \vec{B}_0 e^{-i(\omega t - \vec{k}\cdot\vec{r})} \qquad \vec{B} = \sqrt{\mu\varepsilon}\,\hat{k}\times\vec{E}$$

$$v = \frac{\omega}{k} = \frac{1}{\sqrt{\varepsilon\mu}} \qquad n = \frac{\sqrt{\mu\varepsilon}}{\sqrt{\mu_0\varepsilon_0}} \qquad k = \frac{2\pi}{\lambda} \qquad \theta_c = \sin^{-1}\left(\frac{n'}{n}\right)$$

TE waves: $\vec{\nabla}_t B_z = -\dfrac{ik_c^2}{k_g}\vec{B}_t \qquad \vec{B}_t = \dfrac{k_g}{\omega}\left(\hat{z}\times\vec{E}_t\right) \qquad k_0^2 = k_c^2 + k_g^2$

TM waves: $\vec{\nabla}_t E_z = -\dfrac{ik_c^2}{k_g}\vec{E}_t \qquad \vec{E}_t = -\dfrac{k_g}{\mu\varepsilon\omega}\left(\hat{z}\times\vec{B}_t\right)$

$$\vec{p} = \int \vec{x}'\,\rho(\vec{x}')d^3x' \qquad \vec{A}_{ED}(\vec{x}) = -\frac{i\mu_0\omega\vec{p}}{4\pi}\frac{e^{ikr}}{r} \qquad \vec{B}_{ED} = \frac{\mu_0 ck^2}{4\pi}\frac{e^{ikr}}{r}\hat{n}\times\vec{p}$$

$$\vec{E}_{ED} = -\frac{k^2}{4\pi\varepsilon_0}\frac{e^{ikr}}{r}\left[\hat{n}\times(\hat{n}\times\vec{p})\right] = Z_0\vec{H}_{ED}\times\hat{n} \qquad Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \qquad \frac{dP}{d\Omega} = \frac{c^2k^4Z_0}{32\pi^2}\left|\hat{n}\times\vec{p}\right|^2$$

$$\vec{m} = \frac{1}{2}\int(\vec{x}'\times\vec{J})d^3x' \qquad \vec{B}_{MD} = \frac{\mu_0 k^2}{4\pi}\frac{e^{ikr}}{r}(\hat{n}\times\vec{m})\times\hat{n} \qquad \vec{E}_{MD} = \frac{Z_0k^2}{4\pi}\frac{e^{ikr}}{r}\hat{n}\times\vec{m} \qquad \frac{dP}{d\Omega} = \frac{k^4Z_0}{32\pi^2}\left|\hat{n}\times\vec{m}\right|^2$$

$$\frac{d\sigma}{d\Omega} = \frac{k^4a^6}{2}\left|\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right|^2\left(1+\cos^2\theta\right) \qquad \frac{d\sigma}{d\Omega} = k^4a^6\left[\frac{5}{8}\left(1+\cos^2\theta\right) - \cos\theta\right]$$

$$\frac{d\sigma}{d\Omega} = \frac{k^4a^6}{2}\left|\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right|^2\left(1+\cos^2\theta\right)\left|S(\vec{q})\right|^2 \qquad S(\vec{q}) = \sum_i e^{-i\vec{q}\cdot\vec{x}_i}$$

$$\frac{d\sigma}{d\Omega} = \frac{1+\cos^2\theta}{2}r_0^2 \qquad \sigma = \frac{8\pi}{3}r_0^2$$

$$\begin{pmatrix} x'_0 \\ x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad u'_{\parallel} = \frac{u_{\parallel} - v}{1 - \vec{v}\cdot\vec{u}/c^2} \qquad \vec{u}'_{\perp} = \frac{\vec{u}_{\perp}}{\gamma(1 - \vec{v}\cdot\vec{u}/c^2)}$$

$$\vec{p} = \gamma m_0\vec{u} \qquad E = \gamma m_0 c^2 \qquad p^\mu = (E/c, \vec{p}) \qquad E = \sqrt{m_0^2c^4 + c^2p^2}$$

$$\partial_\mu G^{\mu\nu} = \frac{4\pi}{c}J^\nu \qquad \partial_\mu \wp^{\mu\nu} = 0 \qquad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \qquad \wp^{\mu\nu} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix}$$

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c^3}\left|\dot{\vec{v}}\right|^2\sin^2\theta \qquad P = \frac{2}{3}\frac{q^2}{c^3}\left|\dot{\vec{v}}\right|^2$$

$$\frac{dP(t')}{d\Omega} = \frac{q^2\dot{v}^2}{4\pi c^3}\frac{\sin^2\theta}{(1-\beta\cos\theta)^5} \qquad P = \frac{2}{3}\frac{q^2}{m^2c^3}\left(\frac{dp}{dt}\right)^2$$

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c^3}\frac{\dot{v}^2}{(1-\beta\cos\theta)^3}\left[1-\frac{\sin^2\theta\cos^2\phi}{\gamma^2(1-\beta\cos\theta)^2}\right] \qquad P = \frac{2}{3}\frac{q^2c}{\rho^2}\beta^4\gamma^4$$

$$\Phi(\vec{x},t) = \left[\frac{q}{(1-\hat{n}\cdot\vec{\beta})R}\right]_{ret} \qquad \vec{A}(\vec{x},t) = \left[\frac{q\vec{\beta}}{(1-\hat{n}\cdot\vec{\beta})R}\right]_{ret} \quad t' = t-\frac{R}{c}$$

Fresnel diffraction: $\psi(u,v) = \frac{e^{ikz}}{i\lambda z}e^{i\frac{k}{2z}\left(u^2+v^2\right)}\int_{-\infty}^{\infty}\left[\psi(x,y)e^{i\frac{k}{2z}\left(x^2+y^2\right)}\right]e^{-i\frac{2\pi}{\lambda z}(ux+vy)}dxdy$

Fraunhofer diffraction: $\psi(u,v) = \frac{e^{ikz}}{i\lambda z}e^{i\frac{k}{2z}\left(u^2+v^2\right)}\int_{-\infty}^{\infty}\psi(x,y)e^{-i\frac{2\pi}{\lambda z}(ux+vy)}dxdy$

1. $$\psi(u,v) = \frac{e^{ikz}}{i\lambda z} e^{i\frac{k}{2z}(u^2+v^2)} \int_{-\infty}^{\infty} \psi(x,y)\, e^{-i\frac{2\pi}{\lambda z}(ux+vy)}\, dx\, dy$$

$\psi(x,y) = 1$ for $-\frac{a}{2} < x < \frac{a}{2}$ and $-\frac{b}{2} < y < b/2$

a. $$\int \psi e^{-i\frac{2\pi}{\lambda z}(ux+vy)} dx\,dy = \underbrace{\int_{-a/2}^{a/2} e^{-i\frac{k}{z}xu} dx}\int_{-b/2}^{b/2} e^{-ik/z(vy)} dy$$

$$\left.\frac{e^{-i\frac{k}{z}xu}}{-i\frac{k}{z}u}\right|_{-a/2}^{a/2} = \frac{1}{-i\frac{k}{z}u}\underbrace{\left(e^{-i\frac{k}{z}u\frac{a}{2}} - e^{+i\frac{k}{z}u\frac{a}{2}}\right)}_{-2i\sin\frac{kau}{2z}}$$

$$= \frac{1}{+i\frac{k}{z}u}\,\frac{1}{+i\frac{k}{z}v}\left(+2i\sin\frac{kau}{2z}\right)\left(+2i\sin\left(\frac{kbv}{2z}\right)\right)$$

$$\psi(u,v) = \frac{e^{ikz}}{i\lambda z} e^{i\frac{k}{2z}(\;)} \frac{4}{\left(\frac{k^2}{z^2}uv\right)} \sin\left(\frac{kau}{2z}\right)\sin\left(\frac{kbv}{2z}\right)$$

$$I = |\psi|^2 = I_0\left(\frac{\sin u_a}{u_a}\right)^2\left(\frac{\sin u_b}{u_b}\right)^2 \qquad u_a = \frac{kau}{2z} \qquad u_b = \frac{kbv}{2z}$$

b. max along u:

$$\frac{\partial I}{\partial x} = I_0 2\left(\frac{\sin u_a}{u_a}\right)\left(\frac{u_a\cos u_a - \sin u_a}{u_a^2}\right)\left(\frac{\pi a}{\lambda z}\right)\left(\frac{\sin u_b}{u_b}\right)^2 = 0$$

$\sin u_a\,(u_a\cos u_a - \sin u_a) = 0$ $\Rightarrow \sin u_a = 0$ gives $I = 0$ ← minima

or $\tan u_a = u_a$ $I \neq 0$ ← maxima

Minima at $\boxed{\frac{kau}{2z} = n\pi \qquad u = \frac{\lambda z}{a}}$

Max at $\boxed{\tan\left(\frac{\pi a u}{\lambda z}\right) = \frac{\pi a u}{\lambda z}}$

Similarly along v

c. $\Delta u \sim a$ $\Delta v \sim b$ since they squeezed through aperture

take $\frac{1}{2}$ angular spread to be angle from center to first min.

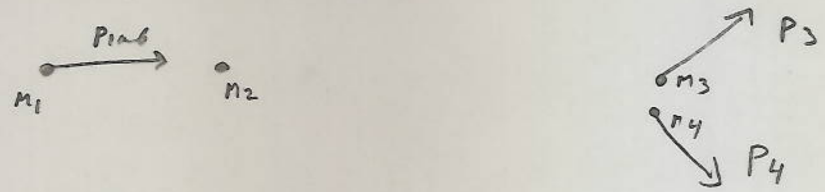
$\boxed{\tan\theta = \lambda/a}$

→ spread in p_x $\Delta p_x = p\sin\theta \simeq \frac{h}{\lambda}\cdot\tan\theta = \frac{h}{\lambda}\frac{\lambda}{a} = \frac{h}{a}$

$$\Delta p_x \Delta x \sim a\frac{h}{a} = h$$

2.

Lab



a.

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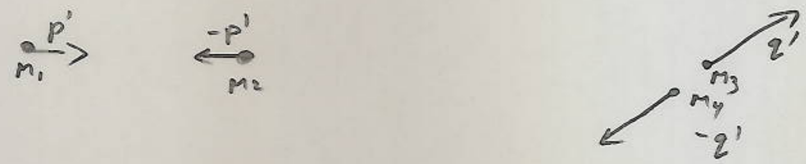

easy to write before

lab $p^\mu = \begin{bmatrix} \frac{E_{lab} + m_2c^2}{c} \\ p_{lab} \\ 0 \\ 0 \end{bmatrix}$ com $p^{\mu\prime} = \begin{bmatrix} \frac{W}{c} \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$p^\mu p_\mu = p'^\mu p'_\mu$: $\left(\frac{E_{lab} + m_2c^2}{c}\right)^2 - p_{lab}^2 = \frac{W^2}{c^2}$

sub in $E_{lab}^2 - p_{lab}^2 c^2 = m_1^2c^4$ for $p_{lab}^2c^2$

$E_{lab}^2 + m_2^2c^4 + 2E_{lab}m_2c^2 - p_{lab}^2c^2 = W^2$

$\cancel{E_{lab}^2} + m_2^2c^4 + 2E_{lab}m_2c^2 + m_1^2c^4 - \cancel{E_{lab}^2} = W^2$

$$\boxed{W^2 = m_2^2c^4 + m_1^2c^4 + 2E_{lab}m_2c^2}$$

b.

Lab

$p_1^\mu = \begin{bmatrix} \frac{E_{lab}}{c} \\ p_{lab} \\ 0 \end{bmatrix}$ $p_2^\mu = \begin{bmatrix} m_2c \\ 0 \\ 0 \\ 0 \end{bmatrix}$ $(p_1^\mu \cdot p_{2\mu})^2 = (m_2E_{lab})^2 = m_2^2(m_1^2c^4 + p_{lab}^2c^2)$

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$p_1'^\mu = \begin{bmatrix} \frac{E_1'}{c} \\ p' \end{bmatrix}$ $p_2'^\mu = \begin{bmatrix} E_2'/c \\ -p' \end{bmatrix}$ $E_{1,2}^2 = m_{1,2}^2c^4 + p^2c^2$

$(p_1'^\mu p'_{2\mu})^2 = \frac{1}{c^4}(E_1'E_2' + p'^2c^2)^2 = \frac{1}{c^4}(\underbrace{E_1'^2E_2'^2}_{\text{sub in}} + p'^4c^4 + 2E_1'E_2'p'^2c^2)$

$= \frac{1}{c^4}\left[(m_1^2c^4 + p'^2c^2)(m_2^2c^4 + p'^2c^2) + p'^4c^4 + 2E_1'E_2'p'^2c^2\right]$

$m_1^2m_2^2c^8 + \underbrace{p'^2c^2(m_2^2c^4 + m_1^2c^4) + 2p'^4c^4 + 2E_1'E_2'p'^2c^2}_{p'^2c^2\left[\underbrace{2p'^2c^2 + m_2^2c^4 + m_1^2c^4}_{E_1^2 + E_2^2} + 2E_1'E_2'\right]}$

$(\underbrace{E_1 + E_2}_{W})^2$

$$= \frac{1}{c^4}\left[m_1^2 m_2^2 c^8 + p'^2 c^2 W^2 \right]$$

Set $\quad p_1'^{\mu} p'_{2\mu} = p_1^{\mu} p_{2\mu}$

$$\cancel{m_1^2 m_2^2 c^4} + p'^2 \frac{W^2}{c^2} = \cancel{m_1^2 m_2^2 c^4} + p_{lab}^2 c^2$$

$$\Rightarrow p'^2 = \frac{p_{lab}^2 c^4}{W^2}$$

since $\vec{p}_{lab}$ and $\vec{p}'$ are parallel,

$$\vec{p}' = \frac{\vec{p}_{lab} c^2}{W}$$

3. $F = -kx + qE_x = m\ddot{x}$ $E_x = E_0 e^{i(kz-\omega t)}$ set $z=0$

$$-\omega_0^2 x + \frac{q}{m}E_x = \ddot{x} \qquad x = x_0 e^{-i\omega t}$$

$$\ddot{x} = -x_0\omega^2 e^{i\omega t}$$

$$-\omega_0^2 x_0 e^{-i\omega t} + \frac{q}{m}E_0 e^{-i\omega t} = -x_0\omega^2 e^{i\omega t}$$

$$x_0(\omega^2-\omega_0^2) + \frac{q}{m}E_0 = 0 \quad \Rightarrow \quad x_0 = \frac{\frac{q}{m}E_0}{\omega^2-\omega_0^2}$$

dipole moment $\vec{p} = q\vec{x}$ $|p| = qx_0$ oscillating at ω

dipole is along $\hat{x}$

$$\frac{dP}{d\Omega} = \frac{c^2k^4 Z_0|p|^2}{32\pi}\underbrace{|\hat{n}\times\hat{x}|^2}$$

$$|\hat{n}\times\hat{x}|^2 = \left|\tfrac{1}{r}(x\hat{x}+y\hat{y}+z\hat{z})\times\hat{x}\right| = \left|\tfrac{1}{r}(y(-\hat{z})+z\hat{y})\right|^2 = \tfrac{1}{r^2}(y^2+z^2) = \sin^2\theta\sin^2\phi+\cos^2\theta$$

$$= 1-\sin^2\theta\cos^2\phi$$

$1-\sin^2\theta$

$1-\sin^2\theta(1-\sin^2\phi)$

$1-\sin^2\theta\cos^2\phi$

$$\frac{d\sigma}{d\Omega} = \frac{1}{P_{inc}}\frac{dP}{d\Omega} \qquad P_{inc} = \frac{c}{2}\varepsilon_0 E_0^2$$

$\frac{Z_0}{\varepsilon c^3}$

$$= \frac{2}{\cancel{c}\varepsilon_0 \cancel{E_0^2}}\frac{c^{\cancel{2}}k^4 Z_0}{\cancel{32}_{16}\pi}\frac{q^2\left(\frac{q}{m}\right)^2\cancel{E_0^2}}{(\omega^2-\omega_0^2)}(1-\sin^2\theta\cos^2\phi)$$

$\sqrt{\frac{\mu}{\varepsilon}}\ \frac{\mu\varepsilon\sqrt{\mu\varepsilon}}{\varepsilon}$

$$= \frac{Z_0\,\omega^4}{\varepsilon_0 c^3\,16\pi}\frac{q^4}{m^2(\omega^2-\omega_0^2)}(1-\sin^2\theta\cos^2\phi)$$

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{\mu^2 q^4\omega^4}{16\pi m^2(\omega^2-\omega_0^2)}(1-\sin^2\theta\cos^2\phi)}$$

$k = \frac{2\pi}{d}$

4. a. $$\boxed{\vec{J}(x,t) = I_0\, \delta(x)\delta(y) \sin(kz)\, e^{-i\omega t}\, \hat{z} \quad \text{for } -\tfrac{d}{2} < z < \tfrac{d}{2}}$$

b. $$\vec{A}(x) = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} I_0 \hat{z} \int \delta(x)\delta(y) \sin(kz')\, e^{-ik\hat{n}\cdot\vec{x}'}\, d^3x'$$

$\hat{n}\cdot\vec{x}' = z'\cos\theta$

$$\int \sin kz'\, e^{-ikz'\cos\theta}\, dz'$$

let $\phi = kz'$, $d\phi = k\,dz'$

$$\frac{1}{k}\int_{-\pi}^{\pi} \sin\phi\, e^{-i\phi\cos\theta}\, d\phi$$

$$I = \int_{-\pi}^{\pi} \underbrace{\sin\phi}_{dv}\, \underbrace{e^{-\alpha\phi}}_{u}\, d\phi \qquad \text{by parts}$$

$$= -e^{-\alpha\phi}\cos\phi\Big|_{-\pi}^{\pi} - \int(-\cos\phi)\, e^{-\alpha\phi}(-\alpha)\, d\phi$$

$$= \left(e^{-\alpha\pi} - e^{+\alpha\pi}\right) \quad -\alpha\int \underbrace{\cos\phi}_{dv}\, \underbrace{e^{-\alpha\phi}}_{u}$$

by parts again

$= -2\sinh\alpha\pi$

$$-\alpha\left[\underbrace{e^{-\alpha\phi}\sin\phi\Big|_{-\pi}^{\pi}}_{0} - \int \sin\phi\, e^{-\alpha\phi}(-\alpha)\, d\phi\right]$$

$$= -2\sinh\alpha\pi - \alpha^2 I$$

$$\Rightarrow \quad I(1+\alpha^2) = -2\sinh\alpha\pi$$

$$I = \frac{-2}{(1+\alpha^2)}\sinh\alpha\pi$$

$\alpha \to i\alpha$

$$I = \frac{-2i}{(1-\alpha^2)}\sin(\alpha\pi)$$

$$= \frac{\mu_0 I_0}{4\pi}\frac{e^{ikr}}{r}\frac{1}{k}\left(\frac{-2i}{1-\cos^2\theta}\sin(\pi\cos\theta)\right)$$

$$\boxed{\vec{A}(x) = -i\frac{\mu_0 I_0}{2\pi}\frac{e^{ikr}}{kr}\frac{\sin(\pi\cos\theta)}{\sin^2\theta}\hat{z}}$$

c. Let $\vec{A} = A\hat{z}$

$$B = \nabla\times\vec{A} \Rightarrow ik\,\hat{n}\times(A\hat{z}) = ikA\,\hat{n}\times\hat{z}$$

From Maxwell $\nabla\times B = \frac{1}{c^2}\frac{\partial E}{\partial t} = -\frac{i\omega}{c^2}E \quad \Rightarrow E = i\frac{c}{k}\nabla\times B$

$$E = i\frac{c}{k}(ik\,\hat{n}\times\vec{B}) = -c\,\hat{n}\times B$$

$$\frac{dP}{d\Omega} = r^2\langle S\cdot n\rangle = \frac{1}{2}r^2\,\hat{n}\cdot(E\times H^*) = \frac{1}{2}r^2\,\hat{n}\cdot\left(-c(\hat{n}\times\vec{B})\times\frac{1}{\mu}\vec{B}\right)$$

$$= \frac{1}{2}r^2\frac{c}{\mu}\,\hat{n}\cdot\left(\vec{B}\times(n\times\vec{B})\right)$$

$$\hat{n}\cdot\left(|B|^2\hat{n} - (B\cdot n)\vec{B}\right) = |B|^2$$

$$= \frac{1}{2}\frac{c}{\mu}i^2k^2|A|^2\underbrace{|n\times\hat{z}|^2}_{\sin^2\theta}$$

$$= \frac{1}{2}\frac{c}{\mu}\cancel{k^2}\frac{\mu^2 I_0^2}{4\pi^2}\frac{1}{k^2}\frac{\sin^2(\pi\cos\theta)}{\sin^{\cancel{4}}\theta}\cancel{\sin^2\theta} = \boxed{\frac{I_0^2}{8\pi^2}\sqrt{\frac{\mu}{\varepsilon}}\frac{\sin^2(\pi\cos\theta)}{\sin^2\theta}}$$

5

$\overset{q\,m}{o\longrightarrow}$ Ze

$$u(r)=\frac{ze^2}{r}\qquad \frac{du}{dr}=-\frac{ze^2}{r^2}=-\frac{u^2}{ze^2}$$

$$P=\frac{2}{3}\frac{e^2}{c^3}|\dot{\vec v}|^2$$

Cons of Energy,

$$\frac{1}{2}mv_0^2=\frac{1}{2}mv^2+u(r)$$

$$u(r)=\frac{1}{2}m\left(v_0^2-v^2\right)$$

$$\Delta E=\int P\,dt$$ ← write in terms of velocity $dv=\dot v\,dt$

$$=\int P\,\frac{dv}{\dot v}$$

$$F=m\dot v=-\frac{\partial u}{\partial r}=\frac{u^2}{ze^2}$$

radiates going in + out

$$=\frac{2}{3}\frac{e^2}{c^3}\;2\underbrace{\int_{v_0}^{0}\frac{\dot v^2}{\dot v}\,dv}$$

$$\int\dot v\,dv=\frac{1}{m\,ze^2}\int u^2\,dv=\frac{1}{m\,Ze^2}\left(\frac{1}{2}m\right)^2\int\left(v_0^2-v^2\right)^2dv$$

$$=\frac{m}{4Ze^2}\int_{v_0}^{0}\left(v_0^4+v^4-2v_0^2v^2\right)dv$$

$$v_0^4(-v_0)+\frac{1}{5}\left(-v_0^5\right)-2v_0^2\frac{1}{3}\left(-v_0^3\right)$$

$$-v_0^5\left(1+\frac{1}{5}-\frac{2}{3}\right)=-\frac{8}{15}v_0^5$$

$15+3-10$

$$=\frac{-2m}{15Ze^2}v_0^5$$

$$=-\frac{8}{45}\frac{m}{Zc^3}v_0^5$$

So its energy when it goes back to ∞ is

$$\frac{1}{2}mv_0^2-\frac{8}{45}\frac{m}{Zc^3}v_0^5=\boxed{\frac{1}{2}mv_0^2\left[1-\frac{16v_0^3}{45Zc^3}\right]}$$

PHYSICS 210B, Winter 2011
Midterm Exam (100 points in total)

- Closed books and closed notes
- Please write down the necessary intermediate steps
- Write your answers in the space provided. Continue on the back if necessary.
- A formula sheet is provided at the end of the exam.

Name:__________________________ ID: ____________________________.

Problem 1: _______________________

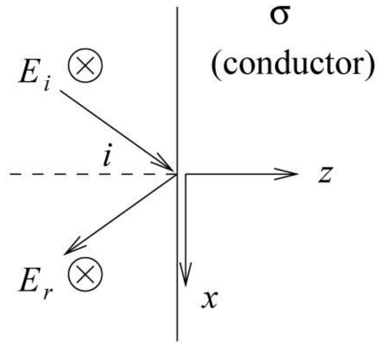
Problem 2: _______________________

Problem 3: _______________________

Total: _______________________

1. (32 points)

A plane polarized electromagnetic wave of frequency ω in free space is incident with angle i on the flat surface of an excellent conductor ($\mu = \mu_0$, $\epsilon = \epsilon_0$ and $\sigma \gg \omega\epsilon_0$) which fills the region $z > 0$.



Consider *only* linear polarization perpendicular to the plane of incidence.

a) If the incident wave is given by $\vec{E} = \vec{E}_i e^{i(\vec{k}\cdot\vec{x}-\omega t)}$, show that (in the limit $\sigma \gg \omega\epsilon_0$) the magnitude of the electric field inside the conductor is

$$E_c = E_i \gamma \cos i \, e^{-z/\delta} e^{i(kx \sin i + z/\delta - \omega t)}$$

where

$$\delta = \sqrt{\frac{2}{\omega\mu_0\sigma}} \qquad \text{and} \qquad \gamma = (1-i)\sqrt{\frac{2\epsilon_0\omega}{\sigma}}$$

The z direction is perpendicular to the flat surface of the conductor, while the x direction is parallel to it.

b) Show that the time averaged power per unit area flowing into the conductor is given by $S^{\perp} = \epsilon_0 |E_i|^2 \omega\delta \cos^2 i$.

You may use the Fresnel equation for E perpendicular to the plane of incidence, $\frac{E_c'}{E_i} = \frac{2}{(1+\frac{\eta_i \cos i}{\eta_c \cos r})}$ where η is the impedance of the material and r is the refracted angle.

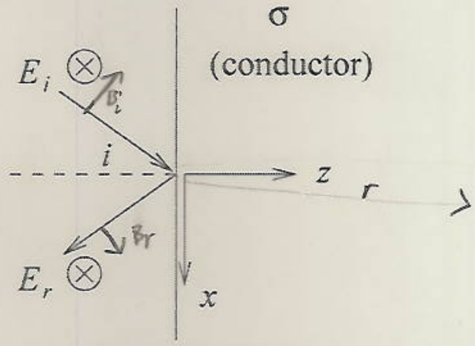
2. An electric dipole oscillates with a frequency ω and amplitude P_0. It is placed at a distance $x = a/2$ from an infinite perfectly conducting grounded plane and the dipole is parallel to the plane. Find the electromagnetic field and the time-averaged angular distribution of the emitted radiation for distances $r >> \lambda >> a$. (32 points)

3. **Propagation of a TE wave between two perfectly conducting plates**. Assume the wave reflects perfectly off each conducting surface, a is the distance between two plates, k is the free-space wave number of the incident plane wave, and θ is the incident angle (the angle between the incident wave and the normal direction to the conducting plate).

(a) Derive the expressions for the cut-off frequency and the wave number along the horizontal (*i.e.* propagation) and the vertical direction. (18 points)

(b) Determine the phase velocity and group velocity of the wave propagating along the plates. (18 points)

1. (32 points)
A plane polarized electromagnetic wave of frequency ω in free space is incident with angle i on the flat surface of an excellent conductor ($\mu = \mu_0$, $\epsilon = \epsilon_0$ and $\sigma \gg \omega\epsilon_0$) which fills the region $z > 0$.



Consider *only* linear polarization perpendicular to the plane of incidence.

a) If the incident wave is given by $\vec{E} = \vec{E}_i e^{i(\vec{k}\cdot\vec{x}-\omega t)}$, show that (in the limit $\sigma \gg \omega\epsilon_0$) the magnitude of the electric field inside the conductor is

$$E_c = E_i \gamma \cos i\, e^{-z/\delta} e^{i(kx\sin i + z/\delta - \omega t)}$$

where

$$\delta = \sqrt{\frac{2}{\omega\mu_0\sigma}} \qquad \text{and} \qquad \gamma = (1-i)\sqrt{\frac{2\epsilon_0\omega}{\sigma}}$$

The z direction is perpendicular to the flat surface of the conductor, while the x direction is parallel to it.

b) Show that the time averaged power per unit area flowing into the conductor is given by $S^{\perp} = \epsilon_0 |E_i|^2 \omega\delta\cos^2 i$.

a.

$$\vec{E}_i = \hat{y} E_0 e^{i(\vec{k}_i\cdot\vec{r})} \qquad \vec{k}_i = k_i\begin{bmatrix}\sin i\\ 0\\ \cos i\end{bmatrix} \qquad \hat{k}_i\times\hat{y} = \begin{bmatrix}-\cos i\\ 0\\ \sin i\end{bmatrix} \qquad \eta_1 = \sqrt{\frac{\mu_0}{\epsilon_0}}$$

$$\vec{H}_i = \frac{1}{\eta_1} E_0 e^{i(\vec{k}_i\cdot\vec{r})}\, \hat{k}_i\times\hat{y}$$

$$\vec{E}_r = \hat{y} E_r e^{i(\vec{k}_r\cdot\vec{r})} \qquad \vec{k}_r = k_r\begin{bmatrix}\sin i\\ 0\\ -\cos i\end{bmatrix} \qquad \hat{k}_r\times\hat{y} = \begin{bmatrix}\cos i\\ 0\\ \sin i\end{bmatrix}$$

$$\vec{H}_r = \frac{1}{\eta_1} E_r e^{i(\vec{k}_r\cdot\vec{r})}\, \hat{k}_r\times\hat{y}$$

$$\vec{E}_c = \hat{y} E_c e^{i(\vec{k}_c\cdot\vec{r})} \qquad \vec{k}_c = k_c\begin{bmatrix}\sin r\\ 0\\ \cos r\end{bmatrix} \qquad \hat{k}_c\times\hat{y} = \begin{bmatrix}-\cos r\\ 0\\ \sin r\end{bmatrix}$$

$$\vec{H}_c = \frac{E_c}{\eta_c} e^{i(\vec{k}_c\cdot\vec{r})}\, \hat{k}_c\times\hat{y}$$

$E_\parallel$ cont: $E_0 + E_r = E_c$

$H_\parallel$ cont: $\frac{1}{\eta_1}(-E_0\cos i + E_r\cos i) = \frac{E_c}{\eta_2}(-\cos r)$

$$E_0 - E_r = \frac{\eta_1}{\eta_2} E_c \frac{\cos r}{\cos i}$$

$$2E_0 = E_c\left(1 + \frac{\eta_1}{\eta_2}\frac{\cos r}{\cos i}\right)$$

$$E_c = \frac{2E_0}{\left(1 + \frac{\eta_1}{\eta_2}\frac{\cos r}{\cos i}\right)}$$

$$\frac{\eta_1}{\eta_2} = \frac{\sqrt{\mu_0/\epsilon_0}}{\sqrt{\mu_0/\epsilon_c}} = \sqrt{\frac{\epsilon_c}{\epsilon_0}}$$

$$\epsilon_c = \epsilon_0 + i\sigma/\omega$$

so $\frac{\eta_1}{\eta_2} = \sqrt{1 + i\sigma/\omega\epsilon_0}$

$$1 + i\sigma/\omega\varepsilon_0 = \sqrt{1 + (\sigma/\omega\varepsilon_0)^2}\ e^{i\phi} \quad \text{where } \phi = \tan^{-1} \sigma/\omega\varepsilon_0$$

we have $\sigma >> \omega\varepsilon_0$ so $\phi = \tan^{-1}\infty = \pi/2$

$$\simeq \frac{\sigma}{\omega\varepsilon_0} e^{i\pi/2}$$

$\sqrt{1+(\sigma/\omega\varepsilon)^2} = \sigma/\omega\varepsilon$

so $$\frac{\eta_1}{\eta_2} = \sqrt{\frac{\sigma}{\omega\varepsilon_0}}\ e^{i\pi/4}$$

Now examine $\dfrac{\cos r}{\cos i}$: from Snell's Law $n \sin i = n_c \sin r$

$\sin r = \frac{n}{n_c} \sin i = \sqrt{\frac{\varepsilon_0}{\varepsilon_c}} \sin i$

$= \frac{\eta_2}{\eta_1} \sin i$

$$\cos r = \sqrt{1 - \sin^2 r}$$
$$= \sqrt{1 - \left(\frac{\eta_2}{\eta_1}\right)^2 \sin^2 i} = \sqrt{1 - \underbrace{\frac{\omega\varepsilon}{\sigma}}_{\text{tiny}} \sin^2 i\, e^{i\pi/2}} \simeq 1$$

so,
$$E_c = \frac{2E_0}{1 + \sqrt{\frac{\sigma}{\omega\varepsilon_0}}\, e^{i\pi/4} \frac{1}{\cos i}} = \frac{2E_0 \cos i}{\underbrace{\cos i}_{\text{small}} + \underbrace{\sqrt{\frac{\sigma}{\omega\varepsilon_0}}\, e^{i\pi/4}}_{\text{big}}} \simeq 2E_0 \cos i \sqrt{\frac{\omega\varepsilon_0}{\sigma}}\ \underbrace{e^{-i\pi/4}}_{\frac{1}{\sqrt{2}}(1-i)}$$

$$\boxed{E_c = \underbrace{\sqrt{\frac{2\omega\varepsilon_0}{\sigma}}\ (1-i)}_{\delta} E_0 \cos i}$$

Now look at $e^{i(\vec{k}_c \cdot \vec{r})}$:

$$\vec{k}_c \cdot \vec{r} = \underbrace{k_c}_{\omega\sqrt{\mu_0\varepsilon_0(1+\frac{i\sigma}{\varepsilon_0\omega})} \simeq \omega\sqrt{\mu_0\varepsilon_0}\sqrt{\frac{\sigma}{\omega\varepsilon_0}}\, e^{i\pi/4}} \Big(\underbrace{\sin r}_{\sqrt{\frac{\omega\varepsilon_0}{\sigma}}\, e^{-i\pi/4} \sin i \text{ from above}}\ x + \underbrace{\cos r}_{\simeq 1}\ z \Big)$$

$$= \omega\sqrt{\mu_0\varepsilon_0}\sqrt{\frac{\sigma}{\omega\varepsilon_0}}\, e^{i\pi/4} \left(\sqrt{\frac{\omega\varepsilon_0}{\sigma}}\, e^{-i\pi/4} \sin i\, x + z \right)$$

$$= \underbrace{\omega\sqrt{\mu_0\varepsilon_0}}_{\omega/c = k} \sin i\, x + \underbrace{\sqrt{\frac{\mu_0\sigma\omega}{2}}}_{1/\delta} (1+i)\, z$$

$$e^{i(\vec{k}_c\cdot\vec{r})} = e^{-z/\delta}\ e^{i(k \sin i\, x + z/\delta)}$$

So finally:

$$\boxed{\vec{E}_c = \hat{y}\, E_0 \gamma \cos i\, e^{-z/\delta}\, e^{i(kx\sin i + z/\delta)}}$$

b. $\langle S \rangle = \frac{1}{2} \mathrm{Re}(E \times H^*)$

At the surface of conductor, take E_c and H_c at $z=0$

Power inwards is

$$\hat{z}\cdot\langle S\rangle = \frac{1}{2}\,\mathrm{Re}\left[E_c \frac{E_c^*}{\eta_c^*}\, e^{-z/\delta} e^{-z/\delta} e^{-i(kx\sin i + z/\delta)} e^{+i(kx\sin i + z/\delta)} \right] \hat{z}\cdot\left(\underbrace{\hat{y}\times(\hat{k_c}\times\hat{y})}\right)$$

$z=0$

$$\hat{z}\cdot\begin{bmatrix}\sin r\\ 0\\ \cos r\end{bmatrix}$$

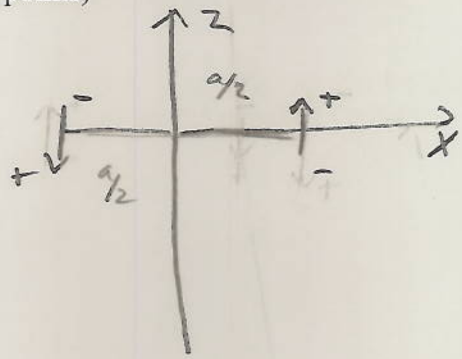
$$= \frac{1}{2}\underbrace{\cos r}_{\simeq 1}\ \mathrm{Re}\left[\frac{2\omega\varepsilon_0}{\sigma}\underbrace{(1-i)(1+i)}_{2} E_0^2\cos^2 i\ \underbrace{\sqrt{\frac{\varepsilon_c^*}{\mu_0}}}_{\sqrt{\frac{\varepsilon_0}{\mu_0}}\sqrt{\frac{\sigma}{\omega\varepsilon_0}}\, e^{-i\pi/4}}\right]$$

$\varepsilon_0 - i\sigma/\omega$

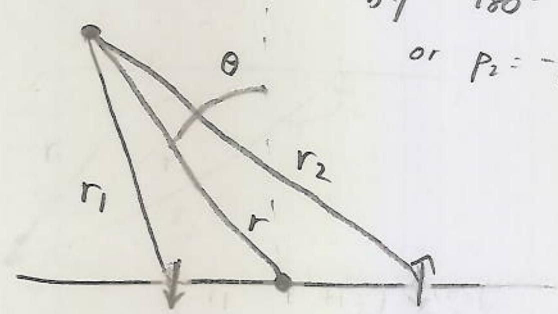
$$= \mathrm{Re}\left[\frac{2\omega\varepsilon_0}{\sigma}\sqrt{\frac{\varepsilon_0}{\mu_0}}\sqrt{\frac{\sigma}{\omega\varepsilon_0}}\frac{1}{\sqrt{2}}(1-i)\ E_0^2\cos^2 i\right]$$

$$= \varepsilon_0 E_0^2\omega\cos^2 i\sqrt{\frac{2}{\sigma\mu_0\omega}} = \boxed{\varepsilon_0 E_0^2\omega\delta\cos^2 i}$$

2. An electric dipole oscillates with a frequency ω and amplitude P_0. It is placed at a distance $x = a/2$ from an infinite perfectly conducting plane and the dipole is parallel to the plane. Find the electromagnetic field and the time-averaged angular distribution of the emitted radiation for distances $r >> \lambda >> a$. (32 points)



2 dipoles out of phase by 180° or $P_2 = -P_1$



Plug Geometry into the vector potential

$$A_{ED}(\vec{x}) = -\frac{i\mu_0}{4\pi}\omega \vec{p}\,\frac{e^{ikr}}{r}$$

$$\vec{A}_{TOT} = \frac{-i\mu_0}{4\pi}\omega P_0\left[\frac{e^{ikr_2}}{r_2} - \frac{e^{ikr_1}}{r_1}\right]\hat{z}$$

minus b.c. dipoles point in opposite directions

$$r_1^2 = r^2 + \left(\tfrac{a}{2}\right)^2 - 2\cdot\frac{ar}{2}\cos\gamma_1$$

$$r_2^2 = r^2 + \left(\tfrac{a}{2}\right)^2 - 2\cdot\frac{ar}{2}\cos\gamma_2$$

$\phi_1 = \pi$, $\theta_1 = \frac{\pi}{2}$

$$\cos\gamma_1 = \cos\theta\cos\theta_1 + \sin\theta\sin\theta_1\cos(\phi - \phi_1)$$

$$= +\sin\theta\cos(\phi - \pi) = -\sin\theta\cos\phi$$

$$\cos\gamma_2 = +\sin\theta\cos\phi$$

$$r_1 = r\left[1 + \left(\frac{a}{2r}\right)^2 + \frac{a}{r}\sin\theta\cos\phi\right]^{1/2}$$

$$\simeq r\left(1 + \frac{a}{2r}\sin\theta\cos\phi\right)$$

$$r_2 \simeq r\left(1 - \frac{a}{2r}\sin\theta\cos\phi\right)$$

$$\frac{1}{r_1} \simeq \frac{1}{r_2} \simeq \frac{1}{r}$$

but need angular dependence in e^{ikr}

$$\vec{A}_{TOT} = \frac{-i\mu_0}{4\pi}\omega P_0\frac{e^{ikr}}{r}\Big[\underbrace{e^{-ik\frac{a}{2}\sin\theta\cos\phi} - e^{ik\frac{a}{2}\sin\theta\cos\phi}}_{-2i\sin\left[k\frac{a}{2}\sin\theta\cos\phi\right]}\Big]\hat{z}$$

$$= \frac{-\mu_0}{2\pi}\omega P_0\frac{e^{ikr}}{r}\frac{ka}{2}\sin\theta\cos\phi\,\hat{z}$$

$\frac{2\pi}{\lambda}\frac{a}{2}$ $\quad \frac{a}{\lambda}$ is small so use $\sin x \simeq x$

$$\vec{A}_{tot} = \frac{-\mu_0}{4\pi} c k^2 a \frac{e^{ikr}}{r} \sin\theta \cos\phi \; \hat{z}$$

$$\hat{z} = (\hat{r}\cos\theta - \hat{\theta}\sin\theta)$$

$$B = \nabla \times \vec{A}$$

$$= \hat{r} \frac{1}{r\sin\theta}\left(-\frac{\partial A_\theta}{\partial \phi}\right) \leftarrow \text{will be of order } \frac{1}{r^2}\text{, ignore}$$

$$+ \hat{\theta}\left[\frac{1}{r\sin\theta} \frac{\partial A_r}{\partial \phi}\right] \leftarrow \text{will also be of order } \frac{1}{r^2}\text{, ignore}$$

same

$$+ \hat{\phi} \underbrace{\frac{1}{r}\left[\frac{\partial}{\partial r}(r A_\theta)\right.}_{\text{only term that will be of order } 1/r} \left. - \frac{\partial A_r}{\partial \theta}\right]$$

$$= \hat{\phi} \frac{1}{r} \frac{\partial}{\partial r}\left(e^{ikr}\right) \cdot \text{constants} = \frac{+\mu_0}{4\pi} c k^2 a \,(ik) \frac{e^{ikr}}{r} (\cancel{+}\sin^2\theta \cos\phi)\, \hat{\phi}$$

$$\boxed{= \frac{i\mu_0}{4\pi} c k^3 a \frac{e^{ikr}}{r} \sin^2\theta \cos\phi \quad \hat{\phi}}$$

$$\vec{E} = \frac{iZ_0}{k} \nabla \times H = \frac{ic}{k} \nabla \times B$$

look at $\nabla \times B = \hat{r}\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta B_\phi)\right] \leftarrow \frac{1}{r^2}$

$+\hat{\theta}\,\frac{-1}{r}\frac{\partial}{\partial r}(r B_\phi)$ ← only keep this

$$= \frac{ic}{k} \frac{i\mu c k^3}{4\pi} a \frac{e^{ikr}}{r} (ik) \sin^2\theta \cos\phi \; \hat{\theta}$$

$$= c\,\vec{B} \times \hat{r} = \boxed{\frac{i\mu_0}{4\pi} c^2 k^3 a \frac{e^{ikr}}{r} \sin^2\theta \cos\phi \; \hat{\theta}}$$

$$\frac{dP}{d\Omega} = r^2 \hat{n} \cdot \langle S \rangle = \frac{1}{2} r^2 \hat{r} \cdot \mathrm{Re}\left(E \times H^*\right) = \frac{1}{2\mu_0}\left(\frac{\mu_0}{4\pi}\right)^2 c^3 k^6 a^2 \sin^4\theta \cos^2\phi \; \hat{r}$$

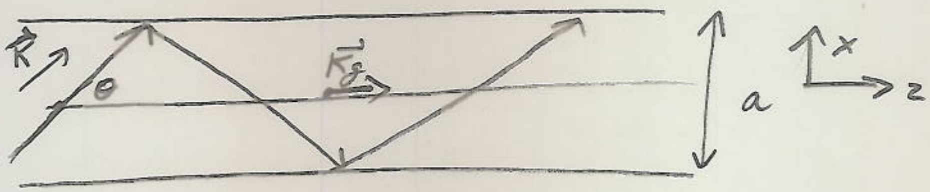
$$\boxed{\frac{dP}{d\Omega} = \frac{\mu_0 c^3 k^6 a^2 \sin^4\theta \cos^2\phi}{32\pi^2}}$$

3. **Propagation of a TE wave between two perfectly conducting plates**. Assume the wave reflects perfectly off each conducting surface, a is the distance between two plates, k is the free-space wave number of the incident plane wave, and θ is the incident angle (the angle between the incident wave and the normal direction to the conducting plate).

(a) Derive the expressions for the cut-off frequency and the wave number along the horizontal (*i.e.* propagation) and the vertical direction. (18 points)

(b) Determine the phase velocity and group velocity of the wave propagating along the plates. (18 points)

a.



A standing wave must build up in the x direction

$a = m\frac{\lambda_x}{2}$ λ is free space wavelength

$\lambda_x = \frac{2a}{m}$ $\Rightarrow$ $K_x = \frac{2\pi}{\lambda_x} = \frac{2\pi}{2a}m = \frac{\pi}{a}m$

$\omega_m^x = K_x c = \frac{c\pi}{a}m$

From geometry,

incident plane wave → $\vec{K} = K_x\hat{x} + K_g\hat{z}$ ← waveguide

only certain angles θ will allow the standing wave pattern to form.

$\vec{K}\cdot\hat{z} = K_g = |K|\cos\theta$

$\Rightarrow K^2 = K_x^2 + K_g^2$

or $K_g = \sqrt{K^2 - K_x^2}$

$$\boxed{\cos\theta = \frac{K_g}{K} = \sqrt{1 - \left(\frac{K_x}{K}\right)^2} = \sqrt{1 - \left(\frac{\omega_m}{\omega}\right)^2}}$$

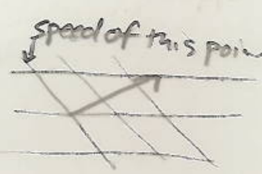
So the cutoff frequency is given by K_g being real, or $K^2 = K_x^2$

$$\boxed{\omega_m = \frac{c\pi}{a}m}$$

b. the group velocity is the speed of the wave in the z direction

$V_g = c\cos\theta = c\sqrt{1 - \left(\frac{\omega_m}{\omega}\right)^2}$

$V_p = \frac{c^2}{V_g} = \frac{c}{\sqrt{1 - \left(\frac{\omega_m}{\omega}\right)^2}}$ or use →



$= \frac{c}{\cos\theta}$

PHYSICS 210A, Winter 2012
Midterm Exam (50 points in total)

- Closed books and closed notes
- Please write down the necessary intermediate steps
- Write your answers in the space provided. Continue on the back if necessary.
- A formula sheet is provided at the end of the exam.

Name:___________________________ ID: _____________________________.

Problem 1: ________________________

Problem 2: ________________________

Problem 3: ________________________

Problem 4: ________________________

Total: ________________________

1. A small sphere of polarizability α and radius a is placed at a great distance from a conducting sphere of radius b, which is maintained at a potential V. For an approximate expression for the force on the dielectric sphere valid for $r >> a$. (10 points; this is a comprehensive exam problem in Fall 2011)

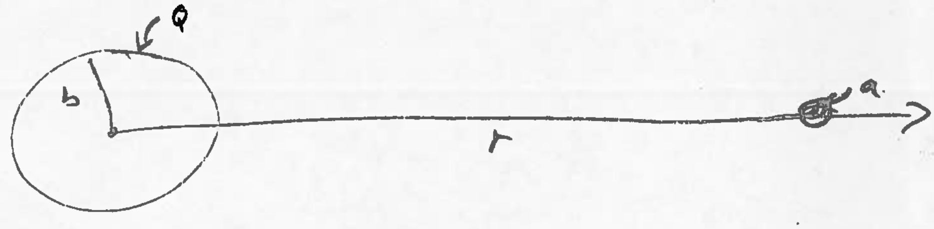
2. Suppose the entire region below plane z = 0 is filled with a uniform and linear dielectric material with permitivity ε. A point charge q is placed a distance d above the origin.

a) Find the potential with $z > 0$. (10 pints)
b) Find the bound charge on the surface of the dielectric material. (4 points)

3. Two infinite thin plates are located at $z = \pm d/2$ with potential $\pm V\cos(ky)$, respectively. Find the electrostatic potential and the electric field in the space between the two plates. (12 points)

4. The linear charge density on a ring of radius a is given by $\rho = \frac{q}{a}(\cos\phi - \sin\phi)$. Find the first three monopole, dipole, and quadrupole moments of the system, and use these moments to write an expression for the potential at an arbitrary point in space. (14 points)

1. A small sphere of polarizability α and radius a is placed at a great distance from a conducting sphere of radius b, which is maintained at a potential V. For an approximate expression for the force on the dielectric sphere valid for $r >> a$. (10 points; this is a comps problem in Fall 2011)



dipole moment $\vec{p} = \varepsilon_0 \alpha \vec{E}$

$$V = \frac{1}{4\pi\varepsilon_0}\frac{Q}{b}$$

$$Q = 4\pi\varepsilon_0 b V$$

E at sphere is

$$E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2} = bV\frac{1}{r^2}$$

induced dipole moment:

$$\vec{p} = \varepsilon_0 \alpha \vec{E} = \frac{\varepsilon_0 \alpha b V}{r^2}\hat{r}$$

Field of dipole: (origin moved to dipole)

$$\vec{E}_{dipole} = \frac{1}{4\pi\varepsilon_0}\frac{3\hat{r}(\hat{r}\cdot\vec{p}) - \vec{p}}{r^3}$$

Align dipole w/ z axis

the dipole field at the conducting sphere is

$\vec{p} = |p|\hat{z}$

$\hat{r} = -\hat{z}$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{-\hat{z}(-\hat{z}\cdot\hat{z})3|p| - |p|\hat{z}}{r^3} = \frac{2|p|}{4\pi\varepsilon_0}\frac{+\hat{z}}{r^3}$$

this acts on a "point charge" Q

$$\vec{F} = Q\cdot\vec{E} = \cancel{4\pi\varepsilon_0} bV \cdot \frac{2}{\cancel{4\pi\varepsilon_0}} \cdot \frac{\varepsilon_0 \alpha b V}{r^2}\frac{\hat{z}}{r^3}$$

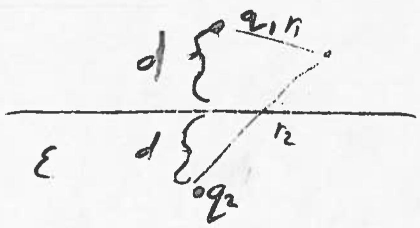
$$\boxed{\vec{F} = \frac{2\alpha b^2 \varepsilon_0 V^2}{r^5}\hat{z}}$$ (attractive)

2. Suppose the entire region below plane $z = 0$ is filled with a uniform and linear dielectric material with permitivity ε. A point charge q is placed a distance d above the origin.

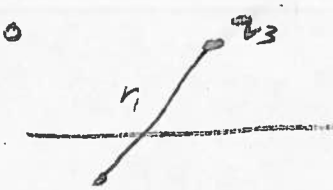
a) Find the potential with $z > 0$. (10 pints)

b) Find the bound charge on the surface of the dielectric material. (4 points)

$z > 0$



$z < 0$



a. $\displaystyle \phi_{z>0} = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1}{r_1} + \frac{q_2}{r_2}\right]$ $\qquad \displaystyle \phi_{z<0} = \frac{1}{4\pi\varepsilon_0}\frac{q_3}{r_1}$

$$\frac{1}{r_{1,2}} = \frac{1}{\left[\rho^2 + (z \mp d)^2\right]^{1/2}}$$

Boundary conditions $\quad E_{\parallel}$ continuous or $\phi_1 = \phi$

$D_{\perp}$ continuous

$$\frac{\partial}{\partial z}\left(\frac{1}{r_{1,2}}\right) = -\frac{1}{2}\,\frac{2(z \mp d)}{\left[\rho^2 + (z \mp d)^2\right]^{3/2}}$$

at $z = 0$ $\quad \displaystyle = \frac{\pm d}{\left[\rho^2 + d^2\right]^{3/2}}$

ϕ continuous: $\quad \displaystyle \frac{q_1}{\left[\rho^2 + d^2\right]^{1/2}} + \frac{q_2}{\left[\rho^2 + d^2\right]^{1/2}} = \frac{q_3}{\left[\rho^2 + d^2\right]^{1/2}}$ $\qquad q_1 + q_2 = q_3$

$D_{\perp}$: $\quad \displaystyle \varepsilon_0 \left.\frac{\partial \phi_{z>0}}{\partial z}\right|_{z=0} = \varepsilon \left.\frac{\partial \phi_{z<0}}{\partial z}\right|_{z=0}$

$$q_1 \frac{d}{\left[\rho^2 + d^2\right]^{3/2}} - q_2 \frac{d}{\left[\rho^2 + d^2\right]^{3/2}} = \frac{\varepsilon}{\varepsilon_0} q_3 \frac{d}{\left[\rho^2 + d^2\right]^{3/2}}$$ $\qquad q_1 - q_2 = \frac{\varepsilon}{\varepsilon_0} q_3$

$2q_1 = (1 + \varepsilon/\varepsilon_0)\, q_3 \quad \Rightarrow$

$2q_2 = (1 - \varepsilon/\varepsilon_0)\, q_3$

$$\boxed{q_3 = \frac{2}{1 + \varepsilon/\varepsilon_0}\, q_1 \qquad q_2 = \frac{(1 - \varepsilon/\varepsilon_0)}{(1 + \varepsilon/\varepsilon_0)}\, q_1}$$

$$\phi_{z>0} = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\left[\rho^2+(z-d)^2\right]^{1/2}} - \frac{\left(\frac{\varepsilon}{\varepsilon_0}-1\right)}{\left(\frac{\varepsilon}{\varepsilon_0}+1\right)}\frac{1}{\left[\rho^2+(z+d)^2\right]^{1/2}}\right]$$

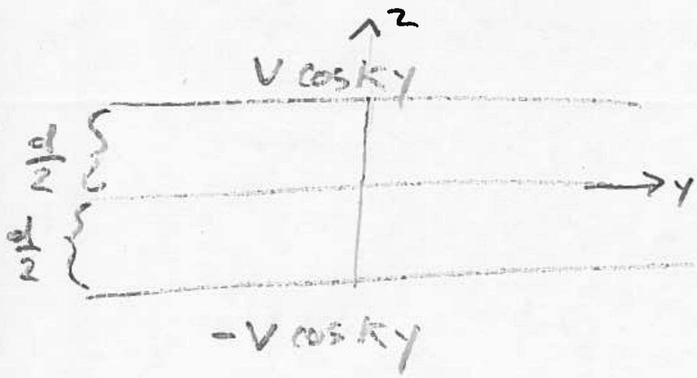
$$\phi_{z<0} = \frac{q}{4\pi\varepsilon_0}\cdot\frac{2}{\left(\frac{\varepsilon}{\varepsilon_0}+1\right)}\frac{1}{\left[(\rho^2+(z+d)^2\right]^{1/2}}$$

b. $\sigma = +P_{z<0}\cdot\hat{z}$ $\qquad P = \varepsilon_0\chi E = \varepsilon_0\cdot\left(\frac{\varepsilon}{\varepsilon_0}-1\right)E$

$$E_z\Big|_{z=0} = \frac{-q}{2\pi\varepsilon_0}\;\frac{1}{\left(\frac{\varepsilon}{\varepsilon_0}+1\right)}\;\frac{-d}{\left[\rho^2+d^2\right]^{3/2}}$$

$$\sigma = \frac{-q}{2\pi}\left(\frac{\varepsilon/\varepsilon_0-1}{\varepsilon/\varepsilon_0+1}\right)\frac{d}{\left[\rho^2+d^2\right]^{3/2}}$$

3. Two infinite thin plates are located at $z = \pm d/2$ with potential $\pm V\cos(ky)$, respectively. Find the electrostatic potential and the electric field in the space between the two plates. (12 points)



$$\phi(z,y) = \sum_k A_k \cos ky \sinh kz$$

$$\phi\left(\pm\tfrac{d}{2}, y\right) = \pm V\cos ky = A\cos ky \sinh k\left(\pm\tfrac{d}{2}\right)$$

$$A = \frac{V}{\sinh kd/2}$$

$$\boxed{\phi = \frac{V}{\sinh k\,d/2}\cos ky \sinh(kz)}$$

4. The linear charge density on a ring of radius a is given by $\rho = \frac{q}{a}(\cos\phi - \sin\phi)$. Find the first three monopole, dipole, and quadrupole moments of the system, and use these moments to write an expression for the potential at an arbitrary point in space. (14 points)

$$\rho = \frac{q}{a}(\cos\phi - \sin\phi)\,\delta(r-a)\,\delta(z)$$

monopole

$$q_0 = \int \rho\, dV = \frac{q}{a}\cdot a \int_0^{2\pi} (\cos\phi - \sin\phi)\, d\phi = 0$$

dipole

$$\vec{p} = \int \vec{x}'\rho\, dv' = \frac{q}{a}\int \begin{bmatrix} r\cos\phi \\ r\sin\phi \\ z \end{bmatrix} (\cos\phi - \sin\phi)\,\delta(r-a)\,\delta(z)\, r\,dr\,d\phi\,dz$$

$$= \frac{q}{a}\int \begin{bmatrix} r^2(\cos^2\phi - \sin\phi\cos\phi) \\ r^2(\sin\phi\cos\phi - \sin^2\phi) \\ 0 \end{bmatrix} \delta(r-a)\, d\phi$$

$$= qa \int \begin{bmatrix} \cos^2 \\ -\sin^2 \\ 0 \end{bmatrix} d\phi$$

$$\vec{p} = qa\pi \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

spherical $r^2 = r^2 + z^2$ cylindrical

quadrupole

$$Q_{ij} = \int (3x_i x_j - r^2 \delta_{ij})\rho\, dv$$

$$Q_{xx} = \frac{q}{a}\int \left(3r^2\cos^2\phi - (r^2 + z^2)\right)(\cos\phi - \sin\phi)\,\delta(r-a)\,\delta(z)\, r\,dr\,d\phi\,dz$$

$$= qa^2 \int (3\cos^2\phi - 1)(\cos\phi - \sin\phi)\, d\phi$$

$$= 0$$

$$Q_{yy} = q/a \int (3r^2\sin^2\phi - (r^2+z^2))(\cos\phi - \sin\phi)\,\delta(r-a)\,\delta(z)\, r\,dr\,d\phi\,dz$$

$$= 0$$

$$Q_{zz} = \frac{q}{a}\int (3\cdot z^2 - (r^2+z^2))(\cos\phi - \sin\phi)\,\delta(r-a)\,\delta(z)\, r\,dr\,d\phi\,dz$$

$$= \frac{q}{a}\int -r^3(\cos\phi - \sin\phi)\,\delta(r-a)\, d\phi$$

$$= 0$$

$$Q_{xy} = \frac{q}{a}\int 3r^2\cos\phi\sin\phi\,(\cos\phi - \sin\phi)\,\delta(r-a)\,\delta(z)\, r\,dr\,d\phi\,dz$$

$$= 3qa^2 \int (\cos^2\phi\sin\phi - \cos\phi\sin^2\phi)\, d\phi$$

$$\boxed{Q_{ij} = 0}$$

$$Q_{xz} = Q_{yz} = 0$$

$$\Phi = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2} = \frac{qa}{4\varepsilon_0}\frac{(\cos\phi - \sin\phi)\sin\theta}{r^2}$$

PHYSICS 210A, Winter 2012
Final Exam (100 points in total)

- Closed books and closed notes
- Please write down the necessary intermediate steps
- Write your answers in the space provided. Continue on the back if necessary.
- A formula sheet is provided in the last page of the exam.

Name:____________________________ ID: ______________________________.

Problem 1: ________________________

Problem 2: ________________________

Problem 3: ________________________

Problem 4: ________________________

Problem 5: ________________________

Problem 6: ________________________

Total: ________________________

1. A sphere of radius R_1 has a charge density ρ uniform within its volume, except for a small spherical hollow region of radius R_2 located a distance a from the center.

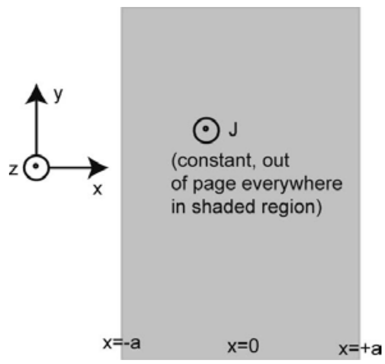
   (a). Find the electric field $\vec{E}$ at the center of the hollow sphere. (7 points)

   (b). Find the potential ϕ at the same point. (7 points)

2. A uniform current density $\vec{J} = \vec{J}_0\hat{z}$ flows through all space between x = - a and x = a (a current sheet), as shown in the figure below.

(a). Find the magnetic field $\vec{B}$ (magnitude and direction) everywhere. (7 points)

(b). An electron (mass m_e, charge $-e$) is fired from an electron gun at x = $2a$ with velocity $\vec{v} = -v\hat{x}$ (toward the origin). What is the minimum speed v the electron must have to reach the point $x = a$ (the edge of the current sheet)? (7 points)



3. A sphere of radius a made of linear magnetic material with permeability μ is placed in an otherwise uniform magnetic field $\vec{H}_0 = H_0\hat{z}$ in vacuum.

(a). Find the magnetic fields, $\vec{H}$, inside and outside the sphere. (10 points)

(b). Find the induced magnetization. (4 points)

(c). Find the bound currents inside the sphere, $\vec{J}_b$, and on its surface, $\vec{K}_b$. (4 points)

4. An infinite straight wire carries a linearly increasing current *I(t)=k t*, for t > 0, where *k* is a constant. Find the electric and magnetic fields ($\vec{E}$ and $\vec{B}$) generated, for $t_r > 0$. (You may ignore delta function pulses associated with the turn-on.) Hint: $\int \frac{dx}{\sqrt{1+x^2}} = \sinh^{-1} x$. (14 points)

5. A cylinder of radius R and infinite length is made of permanently polarized dielectric. The polarization vector $\vec{P}$ is everywhere proportional to the radial vector $\vec{s}$ in a cylinder coordinates (s, ϕ, z), $\vec{P} = a\,\vec{s}$, where a is a positive constant. The cylinder rotates around its axis with an angular velocity ω. This is a non-relativistic problem, -$\omega R << c$.

(a). Find the electric field $\vec{E}$ at a radius s both inside and outside the cylinder. (6 points)

(b). Find the magnetic field $\vec{B}$ at a radius s both inside and outside the cylinder. (6 points)

(c). What is the total electromagnetic energy stored per unit length of the cylinder; (8 points)

(i) before the cylinder started spinning?

(ii) while it is spinning?

Where did the extra energy come from?

6. We assume the existence of magnetic charge related to the magnetic field by the local reaction $\vec{\nabla} \cdot \vec{B} = \mu_0 \rho_m$.

(a). Using the Gauss's theorem, obtain the magnetic field $\vec{B}$ of a point magnetic charge at the origin. (5 points)

(b). In the absence of the magnetic charge, the curl of the electric field is given by the Faraday's law, $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$. Show that this law is incompatible with the magnetic charge density that is a function of time. (5 points)

(c). Assuming that magnetic charge is conserved, derive the local relation between the magnetic charge current density $\vec{J}_m$ and the magnetic density ρ_m. (5 points)

(d). Modify Faraday's law as given in part (b) to obtain a law consistent with the presence of the magnetic charge density that is a function of position and time. (5 points)

Formula Sheet for the Final

$$\rho_b = -\vec{\nabla}\cdot\vec{P} \qquad \sigma_b = \vec{P}\cdot\hat{n}$$

$$\Phi_b(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho_b(\vec{x}')}{|\vec{x}-\vec{x}'|}dV' + \oint_S \frac{\sigma_b}{|\vec{x}-\vec{x}'|}da'$$

$$\vec{J}_b = \vec{\nabla}\times\vec{M} \qquad \vec{K}_b = \vec{M}\times\vec{n}$$

$$\vec{A}_b(\vec{x}) = \frac{\mu_0}{4\pi}\int \frac{J_b(\vec{x}')}{|\vec{x}-\vec{x}'|}dV' + \frac{\mu_0}{4\pi}\int \frac{K_b(\vec{x}')}{|\vec{x}-\vec{x}'|}da'$$

$$\Phi(r,\theta) = \sum_{l=0}^{\infty}\left[A_l r^l + \frac{B_l}{r^{l+1}}\right]P_l(\cos\theta)$$

$$\Phi(r,\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{m}\left[A_l^m r^l + \frac{B_l^m}{r^{l+1}}\right]Y_l^m(\theta,\varphi)$$

$$P_0(x) = 1 \qquad P_1(x) = x \qquad P_2(x) = \frac{1}{2}(3x^2-1) \qquad P_3(x) = \frac{1}{2}(5x^3-3x)$$

$$E_1^{\parallel} = E_2^{\parallel} \qquad D_2^{\perp} - D_1^{\perp} = \sigma_f$$

$$B_1^{\perp} = B_2^{\perp} \qquad \vec{H}_2^{\parallel} - \vec{H}_1^{\parallel} = \vec{K}_f\times\hat{n}$$

$$\vec{p} = \int \vec{x}'\rho(\vec{x}')dV' \qquad \Phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2} \qquad \vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{3\hat{r}\hat{r}\cdot\vec{p}-\vec{p}}{r^3}$$

$$\vec{m} = \frac{1}{2}\int \vec{x}'\times\vec{J}(\vec{x}')dV' \qquad \vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2} \qquad \vec{B}(\vec{x}) = \frac{\mu_0}{4\pi}\frac{3\hat{r}\hat{r}\cdot\vec{m}-\vec{m}}{r^3}$$

$$\vec{D} = \varepsilon\vec{E} \qquad \varepsilon = \varepsilon_0(1+\chi_e) \qquad \vec{B} = \mu\vec{H} \qquad \mu = \mu_0(1+\chi_m)$$

$$\Phi(\vec{x},t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{x}',t_r)}{|\vec{x}-\vec{x}'|}dV' \qquad \vec{A}(\vec{x},t) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{x}',t_r)}{|\vec{x}-\vec{x}'|}dV' \qquad t_r = t - \frac{|\vec{x}-\vec{x}'|}{c} = t - \frac{r}{c}$$

$$\int_V (\vec{\nabla}\cdot\vec{v})dV = \oint_S \vec{v}\cdot d\vec{a} \qquad \oint_C \vec{v}\cdot d\vec{l} = \int_S \vec{\nabla}\times\vec{v}\cdot d\vec{a} \qquad \vec{\nabla}\cdot\vec{v} = \frac{1}{s}\frac{\partial}{\partial s}(sv_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial s} + \frac{\partial v_z}{\partial s}$$

$$W = \frac{1}{2}\int \vec{D}\cdot\vec{E}dV \qquad W = \frac{1}{2}\int \vec{B}\cdot\vec{H}dV$$

$$\nabla\cdot\nabla\times A = 0$$

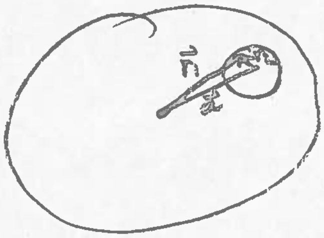
$$\nabla\cdot A = \frac{1}{s}\frac{\partial}{\partial s}(s\,A_s) + \frac{1}{s}\frac{\partial}{\partial\varphi}(A_\varphi) + \frac{\partial}{\partial z}(A_z)$$

$$\nabla\times A = \hat{s}\left(\frac{1}{s}\frac{\partial A_z}{\partial\varphi} - \frac{\partial A_\varphi}{\partial z}\right) + \hat{\varphi}\left(\frac{\partial A_s}{\partial z} - \frac{\partial A_z}{\partial s}\right) + \hat{z}\,\frac{1}{s}\left(\frac{\partial}{\partial s}(s\,A_\varphi) - \frac{\partial A_s}{\partial\varphi}\right)$$

1.

$$4\pi r^2 E = \frac{4}{3}\pi r^3 \rho/\varepsilon_0$$

$$\vec{E} = \frac{\rho}{3\varepsilon_0} r\,\hat{r}$$



$$\vec{r}_1 = \vec{a} + \vec{r}_2$$

a. $$\vec{E} = \frac{\rho}{3\varepsilon_0}\vec{r}_1 - \frac{\rho}{3\varepsilon_0}\vec{r}_2 = \boxed{\frac{\rho}{3\varepsilon_0}\vec{a}}$$, uniform in hole.

b. $$\Phi = \phi_1 + \phi_2$$

Potential in uniform sphere

$$\phi = \frac{Q}{4\pi\varepsilon_0}\frac{1}{R} \quad \text{on surface}$$
$$= \frac{\rho}{3\varepsilon_0} R^2$$

$$\phi = -\int E\,dr$$
$$= -\frac{\rho}{3\varepsilon_0}\int_R^r r'\,dr'$$
$$= -\frac{\rho}{3\varepsilon_0}\left.\frac{r'^2}{2}\right|_R^r$$
$$= \frac{\rho}{6\varepsilon_0}\left(R^2 - r^2\right)$$

$$\phi(r)^{r<R} = \frac{\rho}{3\varepsilon_0}\left[R^2 + \frac{R^2}{2} - \frac{r^2}{2}\right]$$
$$= \frac{\rho}{2\varepsilon_0}\left[R^2 - \frac{r^2}{3}\right]$$

$$\phi_{TOT} = \frac{\rho}{2\varepsilon_0}\left[R_1^2 - \frac{r_1^2}{3} - \left(R_2^2 - \frac{r_2^2}{3}\right)\right]$$

at center of hole, $r_1 = a$
$r_2 = 0$

$$\boxed{\phi_{\text{center of hole}} = \frac{\rho}{2\varepsilon_0}\left[R_1^2 - R_2^2 - \frac{a^2}{3}\right]}$$

2.

Use Ampere's Law

a. $B \cdot L = \mu_0 J_0 \cdot L \cdot a$ $\qquad |x| > a$

$B = \mu_0 a J_0$

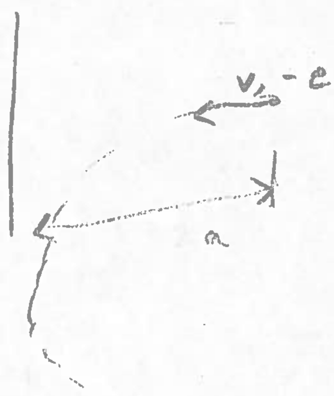
$|x| < a$

$B \cdot L = \mu_0 J L |x|$

$B = \mu_0 J X$

$$\boxed{\vec{B} = \mu_0 J_0 \begin{cases} a \ \hat{y} & x > a \\ X \ \hat{y} & -a > x > a \\ -a \ \hat{y} & x < -a \end{cases}}$$

b.



The electron will curve because its in a uniform B field

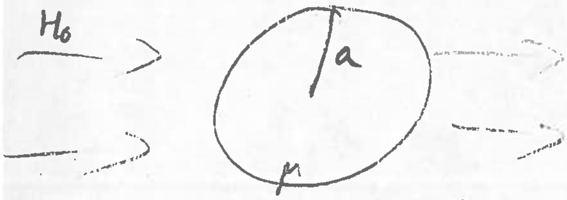
$$F = \frac{mV^2}{R} = eVB$$

$$V = \frac{eBR}{m}$$

we want $R = a$

so $\boxed{V = \frac{eBa}{m} = \frac{\mu_0 e J_0 a^2}{M}}$

3.



$$\Phi_{in} = \sum_l A_l r^l P_l$$

$$\Phi_{out} = \sum_l \frac{B_l}{r^{l+1}} P_l - H_0 r \cos\theta$$

match: $A_l = B_l = 0 \quad l \neq 1$

$$A_1 a = \frac{B_1}{a^2} - H_0 a \qquad \boxed{A_1 = \frac{B_1}{a^3} - H_0}$$

$H = -\nabla\Phi = \frac{1}{\mu} B$

$B_\perp$ cont:

$$\hat{r} \cdot \vec{H}_{in} = -A_1 P_1$$

$$\hat{r} \cdot \vec{H}_{out} = + \left[(+2) \frac{B_1}{r^3} P_1 + H_0 \cos\theta \right]$$

$$-\mu A_1 = \mu_0 \left[\frac{2B_1}{a^3} + H \right] \qquad A_1 = -\frac{\mu_0}{\mu}\left[\frac{2B_1}{a^3} + H_0\right] = \frac{B_1}{a^3} - H_0$$

$$\frac{B_1}{a^3}\left(1 + \frac{2\mu_0}{\mu}\right) = H_0\left(1 - \frac{\mu_0}{\mu}\right)$$

$$B_1 = a^3 H_0 \frac{\mu - \mu_0}{\mu + 2\mu_0}$$

$$\Phi_{in} = H_0\left(\frac{\mu - \mu_0}{\mu + 2\mu_0} - 1\right) r\cos\theta \qquad \Phi_{out} = \frac{B_1}{r^2}\cos\theta - H_0 z$$

$$= -H_0\left(\frac{3\mu_0}{\mu + 2\mu_0}\right) r\cos\theta$$

$$\vec{H}_{in} = H_0 \frac{3\mu_0}{\mu + 2\mu_0} \hat{z} \qquad \vec{H}_{out} = H_0 \hat{z} - B_1\left(\frac{-2\cos\theta}{r^3}\hat{r} - \frac{\sin\theta}{r^3}\hat{\theta}\right)$$

$$= H_0 \hat{z} + H_0 \frac{\mu - \mu_0}{\mu + 2\mu_0} \frac{a^3}{r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

b. $H = \frac{1}{\mu_0} B - M = \frac{1}{\mu} B$

$$M = \left(\frac{1}{\mu_0} - \frac{1}{\mu}\right) B = \left(\frac{\mu - \mu_0}{\mu \mu_0}\right) B = \frac{\mu - \mu_0}{\mu_0} H$$

$$\vec{M} = H_0 \frac{3\mu_0}{\mu + 2\mu_0} \frac{\mu - \mu_0}{\mu_0} \hat{z} = 3 \frac{\mu - \mu_0}{\mu + 2\mu_0} H_0 \hat{z}$$

dipole moment:

$$\phi_{out} = \frac{B_1}{r^2} \cos\theta = H_0 \frac{\mu - \mu_0}{\mu + 2\mu_0} a^3 \frac{\cos\theta}{r^2}$$

$$= \frac{1}{3} |M| a^3 \frac{\cos\theta}{r^2}$$

compare

dipole potential $\vec{A} = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{r}}{r^2}$

or mag potential $\phi_m = \frac{1}{4\pi} \frac{\vec{m} \cdot \hat{r}}{r^2}$

$\frac{4}{3} \pi a^3 M$

$$\vec{m} = \frac{4}{3} \pi a^3 \vec{M}$$

c. $J_b = \nabla \times M$ $\qquad \vec{M} = |M| \hat{z}$

$= 0$

$$\vec{K}_b = M \times \hat{n} = |M| \, \hat{z} \times \hat{r} = |M| \sin\theta \, \hat{\phi}$$

4.

$I = kt \quad t>0$

$\Phi = 0$ (neutral wire)

$t_r = t - \frac{\sqrt{z^2+\rho^2}}{c}$

$\sqrt{\rho^2 + z_{max}^2} = ct$

$z_{max}^2 = c^2t^2 - \rho^2$

$t>0 \quad \vec{A} = \frac{\mu_0}{4\pi} 2 \int_0^{\sqrt{c^2t^2-\rho^2}} \frac{k t_r \, \hat{z}}{\left[\rho^2+z^2\right]^{1/2}} dz$

$$= \frac{\mu_0}{2\pi}\left[\frac{1}{\rho}\int_0^{\sqrt{c^2t^2-\rho^2}} \frac{kt \, dz}{\left[1+\frac{z^2}{\rho^2}\right]^{1/2}} - \frac{k}{c}\int_0^{\sqrt{c^2t^2-\rho^2}} dz\right]$$

$\sinh u = z/\rho \qquad \cosh u \, du = \frac{1}{\rho} dz$

$$\frac{1}{\rho}\int \frac{dz}{\left[1+\frac{z^2}{\rho^2}\right]^{1/2}} = \int \frac{\cosh u \, du}{\left[1+\sinh^2\right]^{1/2}} = \int du = \operatorname{Arcsinh} z/\rho$$

$$\vec{A} = \frac{\mu_0 k}{2\pi}\left[t \operatorname{Arcsinh} \frac{\sqrt{c^2t^2-\rho^2}}{\rho} - \frac{1}{c}\sqrt{c^2t^2-\rho^2}\right] \hat{z} \quad t>0$$

$\sinh y = x$

$\cosh y \, dy = dx$

$\frac{dy}{dx} = \frac{1}{\cosh y}$

$= \frac{1}{\sqrt{1+\sinh^2}}$

$= \frac{1}{\sqrt{1+x^2}}$

$$\vec{E} = -\frac{d\vec{A}}{dt} = -\frac{\mu_0 k}{2\pi}\left[-\frac{1}{\cancel{c}}\frac{1}{\cancel{2}}\frac{\cancel{2}c^{\cancel{2}}t}{\sqrt{c^2t^2-\rho^2}} + \operatorname{Arcsinh}\frac{\sqrt{c^2t^2-\rho^2}}{\rho} + t\left[\underbrace{\frac{1}{\sqrt{1+\frac{c^2t^2-\rho^2}{\rho^2}}}}_{\frac{\rho}{ct}} \cdot \underbrace{\frac{1\cdot \cancel{2}c^2t}{\cancel{2}\rho\sqrt{c^2t^2-\rho^2}}}_{\frac{c^2 t}{\rho\sqrt{c^2t^2-\rho^2}}}\right]\right]$$

$$= -\frac{\mu_0 k}{2\pi}\left[-\frac{ct}{\sqrt{c^2t^2-\rho^2}} + \frac{ct}{\sqrt{c^2t^2-\rho^2}} + \operatorname{Arcsinh}\frac{\sqrt{c^2t^2-\rho^2}}{\rho}\right]$$

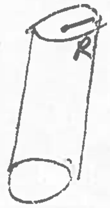
$$= \boxed{-\frac{\mu_0 k}{2\pi}\operatorname{Arcsinh}\sqrt{\frac{c^2t^2}{\rho^2}-1}} \, \hat{z} \quad t_r>0$$

$$B = \nabla\times A = -\frac{\partial A_z}{\partial\rho}\hat{\phi} = -\frac{\mu_0 k}{2\pi}\left[-\frac{1}{c}\frac{1\cdot -\cancel{2}\rho}{\cancel{2}\sqrt{c^2t^2-\rho^2}} + t\frac{1}{\sqrt{1+\frac{c^2t^2}{\rho^2}-1}} \cdot \frac{\cancel{2}\frac{-c^2t^2}{\rho^3}}{\cancel{2}\sqrt{\frac{c^2t^2}{\rho^2}-1}}\right]$$

$$\frac{\rho}{c\sqrt{c^2t^2-\rho^2}} + t\cdot\frac{\rho}{ct}\cdot\frac{-(c^2t^2)}{\rho^2\sqrt{c^2t^2-\rho^2}}$$

$$= -\frac{\mu_0 k}{2\pi}\left[\frac{\rho^2}{\rho c\sqrt{c^2t^2-\rho^2}} - \frac{c^2t^2}{c\rho\sqrt{c^2t^2-\rho^2}}\right] = \boxed{\frac{\mu_0 k}{2\pi c}\frac{\sqrt{c^2t^2-\rho^2}}{\rho}\hat{\phi}}$$

5.



$\vec{P} = a\vec{s}$

$\rho_b = -\nabla \cdot P = \frac{-1}{s}\frac{d}{ds}(s(as))$

$= \frac{-1}{s} a(2s) = -2a$

There is uniform charge density $\rho = -2a$ in cylinder

on surface: $\sigma_b = P \cdot \hat{n} = a s\, \hat{s} \cdot \hat{s} = aR$

a. E in cylinder.
use Gauss's Law

$\cancel{2}\pi \cancel{s} \cdot \cancel{L}\, E = \frac{-\cancel{2}a \cdot \pi s^{\cancel{2}} \cancel{L}}{\varepsilon_0}$

$\boxed{\vec{E} = -\frac{as}{\varepsilon_0}\hat{s}}$ inside

outside

$2\pi s\, L\, E = \frac{-2a\pi R^2 L + aR \cdot 2\pi R L}{\varepsilon_0}$

$= \frac{1}{\varepsilon_0}\left(-aR^2 + aR^2\right) = 0$

$\boxed{E = 0 \text{ outside}}$

b. recall from HW, a moving polarization creates an effective magnetization

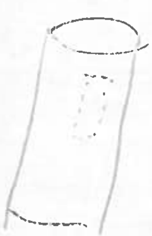
$\vec{M}_{eff} = \cancel{M_{intrinsic}}^{\,0} + \vec{P} \times \vec{V}$ $\qquad \vec{V} = \vec{\omega} \times \vec{s} = s\omega\hat{\phi}$

$= a\omega s^2\, \hat{s} \times \hat{\phi} = a\omega s^2 \hat{z}$

$\vec{J}_b = \nabla \times \vec{M} = -\frac{dM_z}{ds}\hat{\phi} = -2a\omega s\hat{\phi}$ (could have guessed this)

$\vec{K}_b = M \times n = a\omega R^2\, \hat{z} \times \hat{s} = a\omega R^2 \hat{\phi}$

Use Ampere's Law



Inside $s < R$

$B \cdot L = \mu_0(-2a\omega)L\int_0^s s'\,ds' = -2\mu_0 a\omega \frac{s^2}{2}$

$\boxed{\vec{B} = \mu_0 a\omega s^2 \hat{z}}$

outside

$B \cdot L = \mu_0\left[-2a\omega L\frac{R^2}{2} + a\omega R^2 L\right] = 0$

$\boxed{B = 0}$

C. Energy stored

i Before rotation, only electric $W_e = \frac{1}{2}\int D\cdot E = \frac{\varepsilon}{2}\int E^2$

$$\frac{W_e}{L} = \frac{\varepsilon}{2}\frac{a^2}{\varepsilon_0^2}\int_0^R s^2\, s\,ds\,d\phi$$

$$= \frac{2\pi\varepsilon}{2\varepsilon_0^2}a^2\frac{R^4}{4} = \frac{\pi}{4}\frac{\varepsilon}{\varepsilon_0^2}a^2R^4$$

ii With motion, add magnetic

$$\frac{W_B}{L} = \frac{1}{2}\int B\cdot H = \frac{1}{2\mu}\int B^2$$

$$= \frac{1}{2\mu}\mu_0^2 a^2\omega^2\int s^4\, s\,ds\,d\phi$$

$$= \frac{2\pi\mu_0^2}{2\mu}a^2\omega^2\frac{R^6}{6} = \frac{\pi}{6}\frac{\mu_0^2}{\mu}a^2\omega^2R^6$$

$$\frac{\text{Total energy}}{L} = \frac{W_B+W_E}{L} = \frac{\pi}{4}\frac{\varepsilon}{\varepsilon_0^2}a^2R^4 + \frac{\pi}{6}\frac{\mu_0^2}{\mu}a^2\omega^2R^6$$

$$= \frac{\pi\varepsilon}{4\varepsilon_0^2}a^2R^4\left(1+\frac{2}{3}\frac{\varepsilon_0^2\mu_0^2}{\underbrace{\varepsilon\mu}_{\frac{1}{c^2}}}\omega^2R^2\right)$$

The "extra" energy is the magnetic energy that came from the mechanical work needed to get the cylinder rotating

6. $\nabla \cdot B = \mu_0 \rho_m$

a.

$$4\pi r^2 B = \mu_0 q_m$$

$$\boxed{\vec{B} = \frac{\mu_0}{4\pi} \frac{q_m}{r^2} \hat{r}}$$

b.

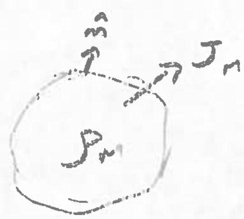
$$\vec{\nabla} \times \vec{E} = -\frac{\partial B}{\partial t}$$

Divergence of both side:
$$\nabla \cdot (\nabla \times E) = -\frac{d}{dt} \nabla \cdot B$$

$$\boxed{0 = \mu_0 \frac{d}{dt} \rho_m}$$

Faraday's law is incompatible with time varying magnetic charge density.

c.



$$\int da \, \vec{J}_m \cdot \hat{n} = -\frac{d}{dt} \int \rho_m \, dV$$

$$\int_V \nabla \cdot J \, dV = -\frac{d}{dt} \int \rho_m \, dV$$

$$\boxed{\nabla \cdot \vec{J}_m = -\frac{d}{dt} \rho_m}$$

d.

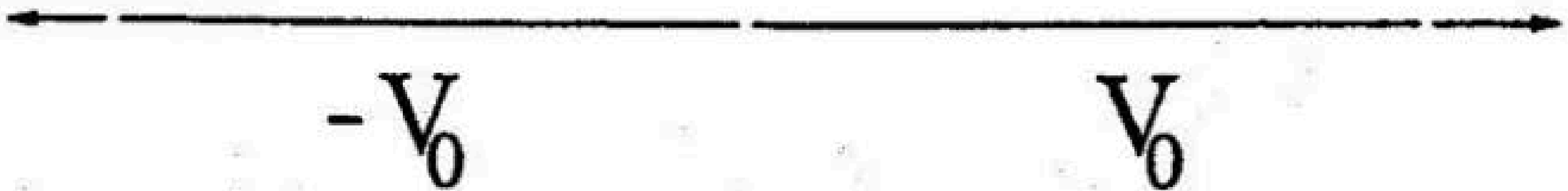
Faradays Law should be consistent with $\nabla \cdot J_m = -\frac{d \rho_m}{dt}$

$$\nabla \cdot (\nabla \times E) = -\frac{d}{dt} \nabla \cdot B - \mu_0 \nabla \cdot J_m$$

$$-\mu_0 \frac{d \rho_m}{dt} + \mu_0 \nabla \cdot J$$

$$\boxed{\nabla \times E = -\frac{dB}{dt} - \mu_0 J_m} \qquad = -\mu_0 \left(\frac{d\rho}{dt} + \nabla \cdot J \right) = 0$$

1. Consider the infinite two-dimensional conducting plane depicted in the figure below. The right half is maintained at electrostatic potential V_0 while the left half is maintained at potential $-V_0$. Calculate the potential above the plane. (25 points)

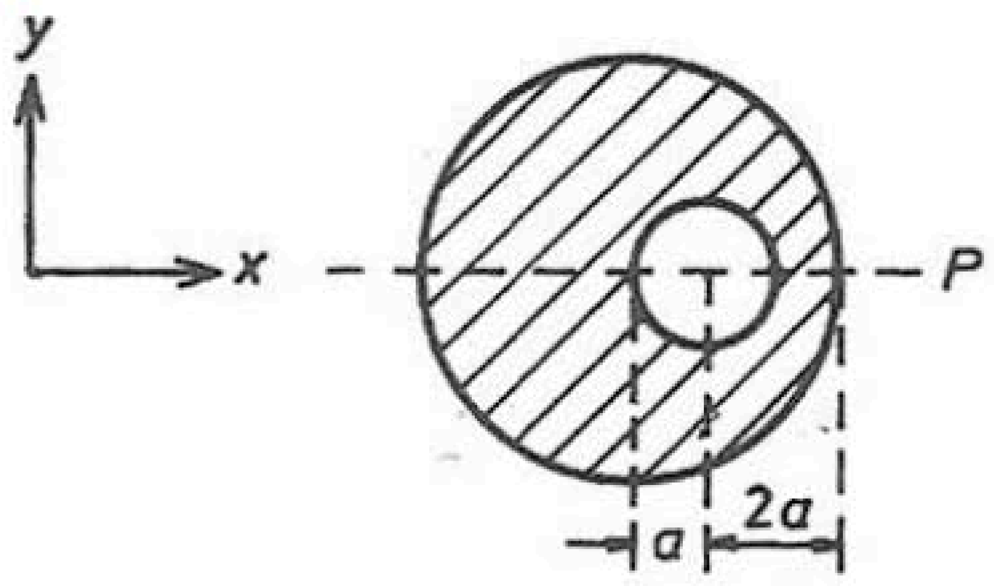


2. The figure below shows the cross section of an infinitely long circular cylinder of radius $3a$ with an infinitely long cylinder hole of radius a displaced so that its center is at a distance a

from the center of the big cylinder. The solid part of the cylinder carries a current I, distributed uniformly over the cross section, and out from the plane of the paper.

(a) Find the magnetic field at all points on the plane P containing the axes of the cylinder. (15 points)

(b) Determine the magnetic field throughout the hole. (10 points)



3. Use the Maxwell's stress tensor to calculate the net force on the northern hemisphere of a uniformly charged solid sphere of radius R and charge q. (25 points)

4. Consider a dielectric medium of infinite extent in all directions. The medium has a tensor permittivity, given by

$$\overleftrightarrow{\varepsilon} = \begin{pmatrix} \varepsilon_{xx} & 0 & 0 \\ 0 & \varepsilon_{yy} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{pmatrix}$$

where $\varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{\perp}$, $\varepsilon_{zz} = \varepsilon_{\parallel}$, $\varepsilon_{\perp} \neq \varepsilon_{\parallel}$ and (x, y, z) refer to Cartesian coordinates. A point charge of charge q is placed at the origin of the coordinate system.

(a) Find the magnitude of the electric field at an arbitrary point (x, y, z), i.e., $\left|\vec{E}\right|$. (10 points)

(b) Deduce the polarization (or bound) charge density ρ_p induced on the dielectric at an arbitrary point (x, y, z). (10 points)

(c) Find the total electrical energy density u_E at (x, y, z). (5 points)

5. A harmonic plane wave of frequency ν is incident normally on an interface between two dielectric media of indices of refraction n_1 and n_2 with $n_2 > n_1$. A fraction α of the energy is reflected and forms a standing wave when combined with the incoming wave. Recall that on reflection the electric field changes phase by π for $n_2 > n_1$ and assume that the z-axis is along the incident wave.

(a) Find the expression for the total electric field as a function of the distance d from the interface in medium n_1. Determine the positions of the maxima and minima of $\langle E^2 \rangle$. (10 points)

(b) Find $B(z, t)$ and $\langle B^2 \rangle$ in medium n_1. (10 points)

(c) When W. Wiener did such an experiment using a photographic plate in 1890, a band of minimum darkening of the plate was found for $d = 0$. Was the darkening caused by the electric or the magnetic field? (5 points)

1. A static charge distribution produces a radial electric field $\vec{E} = A\frac{e^{-br}}{r^2}\hat{r}$, where A and b are constants.

   (a) What is the charge density? (7 points)

   (b) What is the total charge? (5 points)

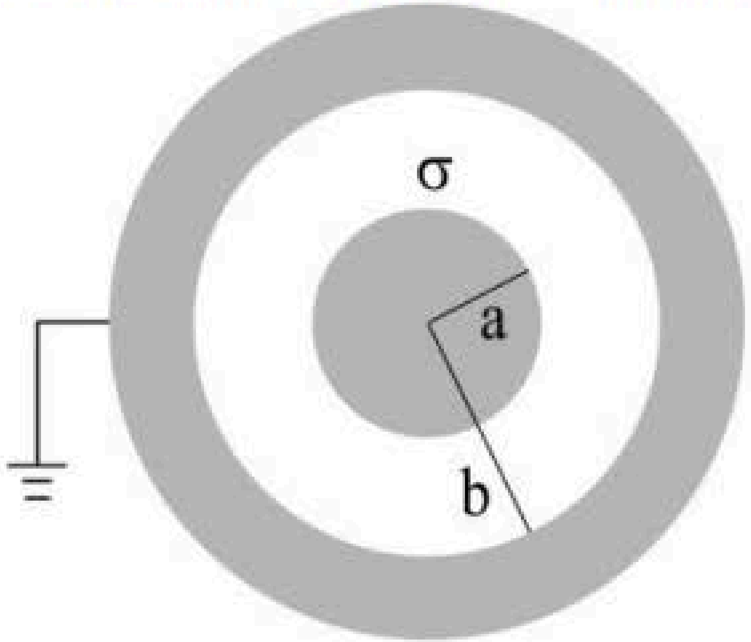
2. A sphere of radius R_1 has a charge density ρ uniform within its volume, except for a small spherical hollow region of radius R_2 located a distance a from the center.

(a). Find the electric field $\vec{E}$ inside the hollow sphere. (6 points)

(b). Find the potential Φ at the center of the hollow sphere. (6 points)

3. Find the potential energy of a point charge (q) in vacuum a distance d from a semi-infinite dielectric medium with a dielectric constant ε_r. (12)

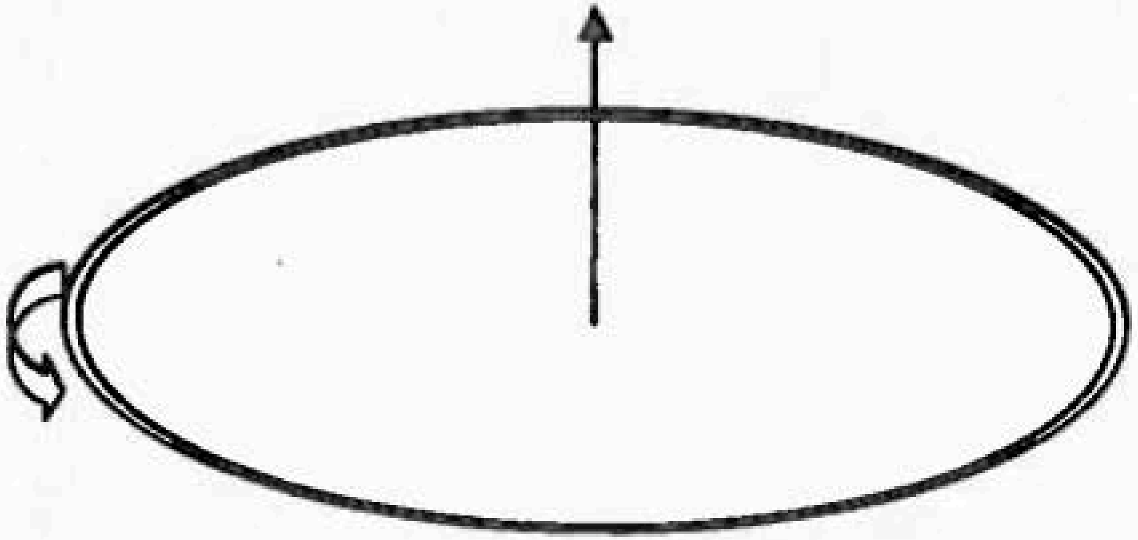
4. An infinitely long cylinder of radius a and surface charge density $\sigma = \sigma_0 \cos 3\varphi$ is surrounded by an infinitely long **conducting** cylindrical tube of inner radius b which is held at zero potential.



(a) Find the potential $\Phi(\rho, \varphi)$ in the $0 \leq \rho < a$ and the $a < \rho \leq b$ regions. (10)

(b) Find the surface charge density on the inner surface of the grounded cylindrical tube. (4 points)

1. A thin uniform donut, carrying charge Q and mass M, rotates about the z-axis as shown in the figure.



(a) Find the ratio of its magnetic dipole moment to its angular momentum. This is called the gyromagnetic ratio (or magnetomechanical ratio). (8 points)
(b) What is the gyromagnetic ratio for a uniform spinning sphere? (4 points)

(c) According to quantum mechanics, the angular momentum of a spinning electron is $\frac{1}{2}\hbar$, where $\hbar$ is Plank's constant. What is the electron's magnetic dipole moment in $A \cdot m^2$? (4 points)
[$e = 1.6 \times 10^{-19} C, m_e = 9.11 \times 10^{-31} kg$ *and* $\hbar = 1.05 \times 10^{-34} Js$].

2. Find the potential energy of a point charge (q) in vacuum a distance d away from a semi-infinite dielectric medium whose dielectric constant is K. (16 points)

3. A cylindrical thin shell of electric charge has length l and radius a, where $l >> a$. The surface charge density on the shell is σ. The shell rotates about its axis with an angular velocity ω which increases slowly with time as $\omega = kt$, where k is a constant and $t \geq 0$, shown below. Neglecting fringing effects, determine:

(a) The magnetic field inside the cylinder. (6 points)
(b) The electric field inside the cylinder. (6 points)
(c) The total electric field energy and the total magnetic field energy inside the cylinder. (5 points)

4. A conducting spherical shell of radius R is cut in half. The two hemispherical pieces are electrically separated from each other but are left close together as shown in the figure below, so that the distance separating the two halves can be neglected. The upper half is maintained at a potential $\phi = \phi_0$, and the lower half is maintained at a potential $\phi = 0$. Calculate the electrostatic potential ϕ at all points in space outside of the surface of the conductors. Neglect terms falling faster than $1/r^4$ (i.e. keep terms up to and including those with $1/r^4$ dependence), where r is the distance from the center of the conductor. [Hint: $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = 3/2\, x^2 - 1/2$, $P_3(x) = 5/2\, x^3 - 3/2\, x$] (17 points)

6. Linearly polarized light of the form $E_x(z,t) = E_0 e^{i(kz-\omega t)}$ is incident normally onto a material which has index of refraction n_R for right-hand circularly polarized light and n_L for left-hand circularly polarized light. Determine the polarization of the reflected light and calculate the reflection coefficient of the intensity. (17 points)

1. Consider a sphere of radius R center at the origin. Suppose a point charge q is put at the origin and that this is the only charge inside or outside the sphere. Furthermore, the potential is $\Phi = V_0\cos\theta$ on the surface of the sphere. What is the electric potential both inside and outside the sphere? (12 points)

2. Consider a dielectric medium of infinite extent in all directions. The medium has a tensor permittivity, given by

$$\overset{\tau}{\varepsilon}=\begin{pmatrix} \varepsilon_{xx} & 0 & 0 \\ 0 & \varepsilon_{yy} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{pmatrix}$$

where $\varepsilon_{xx}=\varepsilon_{yy}=\varepsilon_{\perp}$, $\varepsilon_{zz}=\varepsilon_{\parallel}$, $\varepsilon_{\perp}\neq\varepsilon_{\parallel}$ and (x, y, z) refer to Cartesian coordinates. A point charge of charge q is placed at the origin of the coordinate system.

(a) Find the magnitude of the electric field at an arbitrary point (x, y, z), i.e. $|E|$. (5 points)

(b) Deduce the polarization (or bound) charge density ρ_p induced on the dielectric at an arbitrary point (x, y, z). (5 points)

(c) Find the total electrical energy density u_E at (x, y, z). (3 points) [Hint: $u_E = E\cdot D$]

3. A hydrogen atom is made up of a proton of charge e and an electron of charge $-e$. In 1913 Niels Bohr developed a theoretical model for the hydrogen atom. In Bohr's model, the hydrogen atom is most stable, when electron is at the ground state, *i.e.* the distance between the electron and the proton is equal to the Bohr radius a_0.

(a) The energy to move the electron from the ground state to infinity is called the binding energy. Calculate the binding energy for the hydrogen atom. (5 points) [$a_0 = 0.52$ Å $= 0.52 \times 10^{-10}\ m$; $e = 1.6 \times 10^{-19}\ C$; $\varepsilon_0 = 8.85 \times 10^{-12} \dfrac{C^2}{N \cdot m^2}$; you need to calculate a number.]

(b) Bohr's model is too crude, however. According to the quantum mechanics, the motion of the electron causes its charge to be "smeared out" into a spherical distribution around the proton, so that the electron is equivalent to a charge per unit volume of $\rho(r) = -\dfrac{e}{\pi a_0^3} e^{-2r/a_0}$. Find the total amount of the hydrogen atom's charge that is enclosed within a sphere of radius r centered on the proton. (5 points) [Hint: use integration by parts and keep a_0 and e in the final result.]

(c) Find the electric field caused by the charge of the hydrogen atom as a function of r in (b). (3 points) [keep a_0, ε_0 and e in the final result.]

4. Two concentric conducting spheres of inner and outer radii a and b, respectively, carry charges $\pm Q$. The empty space between the spheres is half-filled by a hemispherical shell of dielectric (of dielectric constant ε_r) as shown in the figure.

(a) Find the electric field everywhere between the spheres. (6 points)

(b) Calculate the surface-charge distribution on the inner sphere. (3 points)

(c) Calculate the bound (or polarization) charge density induced on the surface of the dielectric at $r = a$. (3 points)

1. A thin linear antenna of length d, centered at the origin, and parallel to the z axis, is excited in such a way that the current (I) makes a full wavelength of sinusoidal oscillation at frequency ω.

(a) Find the current density, $\vec{J}(\vec{x},t)$. (3 points)

(b) Find the vector potential of the radiation field, $\vec{A}(\vec{x},t)$, in the far zone. (7 point)

(c) Calculate the power radiated per unit solid angle, $\dfrac{dP}{d\Omega}$, in the far zone. (7 point)

(Hint: if d, λ << r, then $\vec{A}(\vec{x}) = \dfrac{\mu_0}{4\pi}\dfrac{e^{ikr}}{r}\int \vec{J}(\vec{x}\,')e^{-ik\hat{n}\cdot\vec{x}\,'}d^3x\,'$)

2. **Radiating electric quadrupole**. Suppose an oscillating spheroidal distribution of charge with angular frequency ω (a spheroid is an ellipsoid having two axes of equal length).

(a) Assume the length along the x and y axes of the spheroidal distribution to be equal and $Q_{33} = Q_0$. Calculate the other elements of the electric quadrupole moment tensor. (8 points)

(b) Calculate the angular distribution of the radiated power as a function of θ. (8 points)

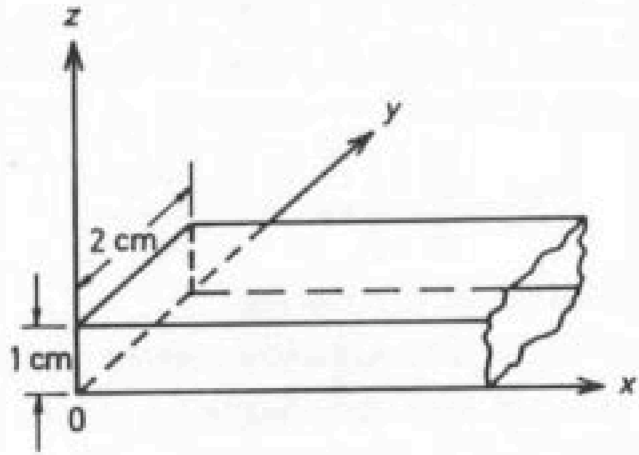
(Hint: $Q_{ij} = \int \left(3x_i' x_j' - x'^2 \delta_{ij}\right) \rho(\vec{x}') d^3x'$ $Q_i(\hat{n}) = \sum_j Q_{ij} n_j$ $\dfrac{dP}{d\Omega} = \dfrac{c^2 k^6 Z_0}{1152\pi^2} \left|\hat{n} \times \vec{Q}(\hat{n})\right|^2$).

3. Consider a rectangular waveguide, infinitely long in the x-direction, with width (y-direction) 2 cm and a height (z-direction) 1 cm. The walls are perfect conductor, shown in Fig. 1.

(a) What are the boundary conditions on the components of $\vec{E}$ and $\vec{B}$ at the walls. (4 points)

(b) Write down the wave equations describing the $\vec{E}$ and $\vec{B}$ fields of the lowest mode. (Hint: the lowest mode has the electric field in the z-direction only). (5 points)

(c) For the lowest mode that can propagate, find the phase velocity and the group velocity. (8 points)

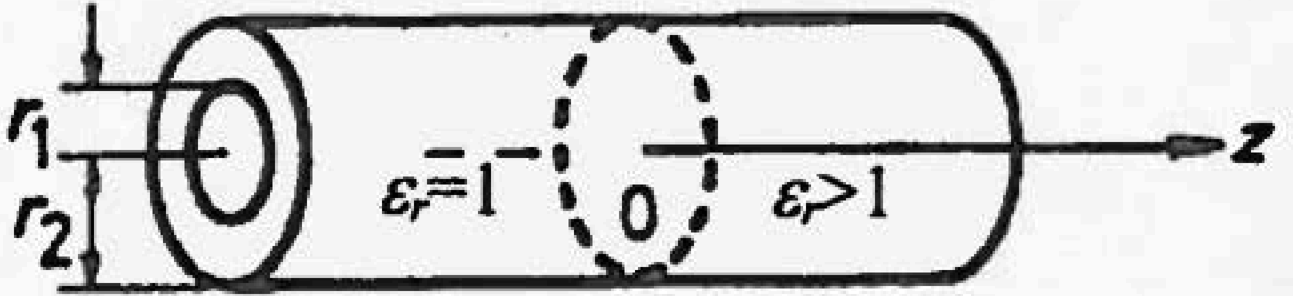


2. Two coaxial cylindrical conductors with radius r_1 and r_2 form a waveguide (shown in the figure below). The region between the conductors is vacuum for $z < 0$ and if filled with a dielectric medium with dielectric constant ε_r for $z > 0$. Assuming $\mu = \mu_0$,

(a) Calculate the E and B field of the TEM mode for $z < 0$ and $z > 0$. (9 points)

(b) If an electromagnetic wave in such a mode is incident from the left on the interface, calculate the transmitted and reflected waves. (5 points)

(c) What fraction of the incident energy is transmitted? What fraction is reflected? (4 points)



3. A small circuit loop of wire of radius a carries a current $i = i_0 cos(\omega t)$. The loop is located in the xy plane (see the figure below) with $r >> a$.

(a) Calculate the first non-zero multiple moment of the system. (5 points)

(b) Calculate the E and B fields of the first non-zero multiple moment.

(c) Determine the angular distribution of the radiation power and qualitatively plot the radiation pattern.

[Hint: you may directly use any the following equations: $\vec{A}_{ED}(\vec{x}) = -\frac{i\mu_0 \omega p}{4\pi}\frac{e^{ikr}}{r}$,

$\vec{A}_{MD}(\vec{x}) = \frac{\mu_0 ik}{4\pi}\frac{e^{ikr}}{r}\hat{n}\times\vec{m}$, $\vec{B}_{EQ} = -\frac{\mu_0 ick^3}{24\pi}\frac{e^{ikr}}{r}\hat{n}\times\vec{Q}(\hat{n})$ and $Q_{ij} = \int (3x_i' x_j' - x'^2 \delta_{ij})\rho(x')d^3x'$. But please write down all the other necessary steps to derive your solutions].

PHYSICS 210B, Spring 2014
Final Exam (100 points in total)

- Closed books and closed notes
- Please write down the necessary intermediate steps
- Write your answers in the space provided. Continue on the back if necessary.

Name: Solutions ID: ________________.

Problem 1: ________________

Problem 2: ________________

Problem 3: ________________

Problem 4: ________________

Problem 5: ________________

Total: ________________

1. A plane electromagnetic wave of frequency ω and wave number k propagates in the $+z$ direction. For $z < 0$, the medium is air with $\varepsilon = \varepsilon_0$ and conductivity $\sigma = 0$. For $z > 0$, the medium is a lossy dielectric with $\varepsilon > \varepsilon_0$ and $\sigma > 0$. Assume that $\mu = \mu_0$ in both media.
   (a) Find the dispersion relation (i.e. the relationship between k and ω) in the lossy medium Please calculate both the real and imaginary part of k. (5 points)
   (b) Find the limiting values of k (both the real and imaginary part) for a very good conductor and a very poor conductor. (4 points)
   (c) Find the e^{-1} penetration depth δ for plane wave power in the lossy medium. (4 points)
   (d) Find the power transmission coefficient T for transmission from $z < 0$ to $z > 0$, assuming $\sigma \ll \varepsilon\omega$ in the lossy medium. (7 points)

a) Maxwell's equations: $\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \Rightarrow i\vec{k}\times\vec{H} = \sigma\vec{E} - i\omega\varepsilon\vec{E}$

assuming harmonic time dependence $\vec{\nabla}\times\vec{E} = -\frac{\partial B}{\partial t} \Rightarrow i\vec{k}\times\vec{E} = i\omega\mu_0 H$

So since $\vec{k}\cdot\vec{E} = 0 \Rightarrow i\frac{(-k^2)}{\omega\mu_0}\vec{E} = \sigma\vec{E} - i\omega\varepsilon\vec{E}$

$$\Rightarrow \boxed{k^2 = \omega^2\varepsilon\mu_0 + i\omega\mu_0\sigma}$$

Assume $k = \alpha + i\beta \Rightarrow k^2 = (\alpha^2 - \beta^2) + 2i\alpha\beta$

So $\alpha^2 - \beta^2 = \omega^2\varepsilon\mu_0$

$2\alpha\beta = \omega\mu_0\sigma$

$$\Rightarrow \boxed{\begin{aligned} \mathrm{Re}\,k = \alpha &= \omega\sqrt{\mu_0\varepsilon}\left[\frac{1}{2}\left(1+\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}\right)\right]^{1/2} \\ \mathrm{Im}\,k = \beta &= \omega\sqrt{\mu_0\varepsilon}\left[\frac{1}{2}\left(-1+\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}\right)\right]^{1/2} \end{aligned}}$$

b) • Good conductor, $\frac{\sigma}{\varepsilon\omega} \gg 1$, so

$$\beta = \alpha \approx \sqrt{\frac{\omega\mu_0\sigma}{2}}$$

• Poor conductor, $\frac{\sigma}{\varepsilon\omega} \ll 1$, so $\beta \approx \frac{\sigma}{2}\sqrt{\frac{\mu_0}{\varepsilon}}$

$\alpha \approx \omega\sqrt{\mu_0\varepsilon}$

c) The transmitted wave is $E_T = E_T e^{-\beta z} e^{i(\alpha z - \omega t)}$

So the e^{-1} penetration depth is $\frac{1}{\beta}$ ← from part (a)

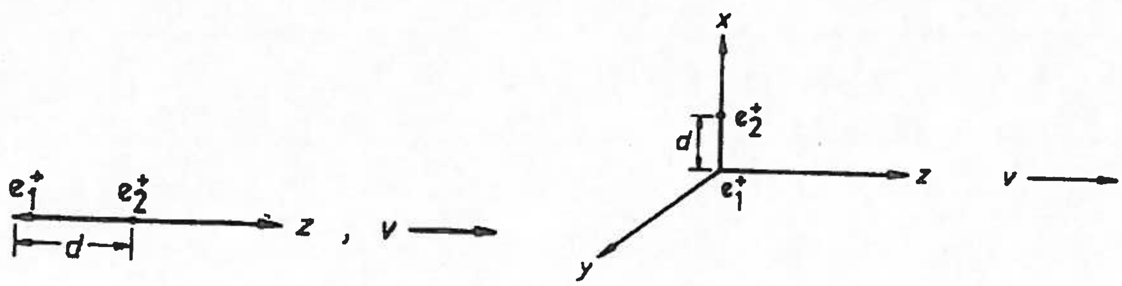
d) Recall that $\frac{E_T}{E_I} = \frac{2}{1+n'}$ and $n' = \frac{ck}{\omega} = \frac{c}{\omega}(\alpha + i\beta)$

So $T = \dfrac{\frac{1}{2}\sqrt{\frac{\varepsilon}{\mu_0}}\,|E_T|^2}{\frac{1}{2}\sqrt{\frac{\varepsilon_0}{\mu_0}}\,|E_I|^2} = \sqrt{\frac{\varepsilon}{\varepsilon_0}}\,\dfrac{4}{|1+n'|^2} = \dfrac{4\sqrt{\varepsilon/\varepsilon_0}}{1+2\,\mathrm{Re}\,n' + |n'|^2}$

$$= \frac{4\sqrt{\varepsilon/\varepsilon_0}}{1 + \frac{2c\alpha}{\omega} + \left(\frac{c}{\omega}\right)^2 (\alpha^2+\beta^2)}$$

Assuming $\sigma \ll \omega\varepsilon$, we have $\boxed{T \approx \dfrac{4\sqrt{\varepsilon/\varepsilon_0}}{\left(1+\sqrt{\varepsilon/\varepsilon_0}\right)^2 + \frac{\varepsilon}{\varepsilon_0}\sigma^2/4\varepsilon^2\omega^2}}$

2. (a) Consider two positrons in a beam at SLAC. The beam has energy of about 50 GeV ($\gamma \approx 10^5$). In the beam (rest) frame, they are separated by a distance d, and positron e_2^+ is traveling directly ahead of e_1^+ in the Z-axis, shown in the figure (left) below. Write down E, B, the Lorentz force F and the acceleration a at e_1^+ due to e_2^+. Do this in both the rest and laboratory frames. (10 points)
(b) The problem is the same as in part (a) except this time the two positrons are traveling side by side as shown in the figure (right) below. (10 points)



Two positrons separated by a distance of d travel with a velocity of v in the Z axis.

a) In the rest frame, K', we have $\vec{E}' = -\frac{e}{4\pi\varepsilon_0 d^2}\hat{z}$
and $\vec{B}' = 0$. $\vec{F}' = -\frac{e^2}{4\pi\varepsilon_0 d^2}\hat{z}$ and $\vec{a}' = -\frac{e^2}{4\pi\varepsilon_0 m d^2}\hat{z}$

In the lab frame, K, we have $\vec{E} = \vec{E}'$, $\vec{B} = \vec{B}' = 0$
$\vec{F} = e\vec{E} = \vec{F}'$ and $\vec{a} = \frac{\vec{a}'}{\gamma^3}$

See notes

b) In k', $\vec{E}' = -\frac{e}{4\pi\varepsilon_0 d^2}\hat{x}$ $\vec{B}' = 0$

$$\vec{F}' = -\frac{e^2}{4\pi\varepsilon_0 d^2}\hat{x} \qquad \vec{a}' = -\frac{e^2}{4\pi m \varepsilon_0 d^2}\hat{x}$$

In k, $\vec{E} = \gamma \vec{E}'$, $\vec{B} = \gamma \frac{\beta}{c} \vec{E}' \hat{y}$ $\vec{F} = \frac{\vec{F}'}{\gamma}$

$$\vec{a} = \frac{F}{m\gamma} = \frac{\vec{a}'}{\gamma^2}$$

3. To account for the effects of energy radiation by an accelerating charge particle, we must modify Newton's equation of motion by adding a radiative reaction force F_R.

(a) Assume for simplicity that the orbit is circular so that $v \cdot \dot{v} = 0$. Show the classic result for F_R is:

$$\vec{F}_R = \frac{2}{3}\frac{e^2}{c^3}\dddot{v}. \qquad \text{(7 points).}$$

(b) Now consider a free electron. Let a plane wave with electric field $E = E_0 e^{-i\omega t}$ be incident on the electron. Again assume that $v \ll c$. What is the time-averaged force $< F >$ on the electron due to the electromagnetic wave? (9 points)

(c) Use the radiation pressure p of this wave to deduce the effective cross section for the scattering of radiation: $\sigma = \langle F \rangle / p$. [Hint: $p = \langle S \rangle / c$] (4 points)

a) Larmor formula says $P = \frac{2e^2}{3c^3}\dot{\vec{v}}\cdot\dot{\vec{v}} = -\vec{F}\cdot\vec{v}$

If we try $\vec{F} = \frac{2}{3}\frac{e^2}{c^3}\ddot{\vec{v}}$, we get

$$\dot{\vec{v}}\cdot\dot{\vec{v}} = -\ddot{\vec{v}}\cdot\vec{v} = -\frac{d}{dt}\underbrace{\cancel{(\vec{v}\cdot\dot{\vec{v}})}}_{0} + \dot{\vec{v}}\cdot\dot{\vec{v}} \checkmark$$

So this works

b) Equation of motion says

$$m\ddot{\vec{r}} = -e\vec{E}_0 e^{-i\omega t} + \frac{2e^2}{3c^3}\dddot{\vec{r}}$$

Solved by $\vec{r} = \vec{r}_0 e^{-i\omega t}$ where $\vec{r}_0 = \dfrac{e\vec{E}_0}{m\omega^2 + i\,\dfrac{2e^2\omega^3}{3c^3}}$

$$\text{So } \langle\vec{F}\rangle = \langle -e\vec{E}_0 e^{-i\omega t} - \frac{e}{c}\vec{v}\times\vec{B}\rangle$$

$$= -\frac{e}{c}\langle\vec{v}\times\vec{B}\rangle \quad \text{where } \langle e^{-i\omega t}\rangle = 0 \text{ and}$$

$$\vec{B} = \hat{k}\times\vec{E}$$

Since $\vec{v} = \dot{\vec{r}} = -i\omega\vec{r}_0 e^{-i\omega t}$, we get

$$\langle \vec F\rangle = -\frac{e}{c}\underbrace{\langle e^{-2i\omega t}\rangle}_{\frac{1}{2}}\ \mathrm{Re}\left[\frac{i\omega e E_0}{m\omega^2 + i\frac{2e^2\omega^3}{3c^3}} \times(\hat k\times \vec E_0)\right]$$

$$= -\frac{e^2}{2c}|E_0|^2\hat k\ \mathrm{Re}\left[\frac{i\omega}{m\omega^2 - i\frac{2e^2\omega^3}{3c^3}}\right]$$

$$= -\frac{e^2}{2c}|E_0|^2\hat k\left(-\frac{2e^2\omega^4}{3c^3}\right)\left[m^2\omega^4+\left(\frac{2e^2\omega^3}{3c^3}\right)^2\right]^{-1}$$

$$\approx \boxed{\frac{e^4|E_0|^2}{3m^2c^4}}\hat k \quad \text{since } \omega r_0 \ll c, \quad r_0 = \frac{e^2}{mc^2}$$

c) $\langle p\rangle = \frac{1}{c}|\langle \vec S\rangle| = \frac{|E_0|^2}{c}\frac{c}{8\pi}$ since $\langle S\rangle = \frac{c}{4\pi}\frac{1}{2}\mathrm{Re}\,(E^*\times B)$

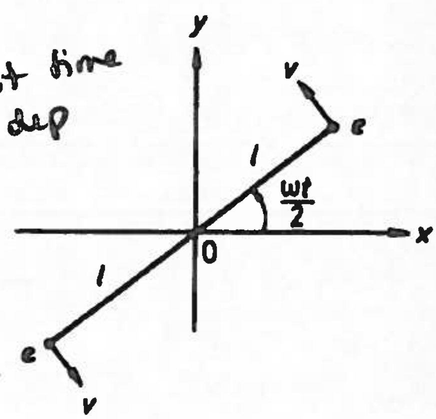
then $\sigma = \frac{|\langle F\rangle|}{\langle p\rangle} = \frac{e^4|E_0|^2}{3m^2c^4}\frac{8\pi}{|E_0|^2} = \boxed{\frac{8\pi}{3}r_0^2}$

4. Two point charges of charge e are located at the ends of a line of length $2l$ that rotates with a constant angular velocity $\omega/2$ about an axis perpendicular to the line and through its center as shown in the figure below.

a) Find (1) the electric dipole moment, (2) the magnetic dipole moment, and (3) the electric quadrupole moment. (10 points)

b) What is the lowest order radiation emitted by this system? Calculate E and B of the radiation. (10 points)

[Hint: $\vec B_{ED} = \frac{\mu_0 ck^2}{4\pi}\frac{e^{ikr}}{r}\hat n\times\vec p$, $\vec B_{MD} = \frac{\mu_0 k^2}{4\pi}\frac{e^{ikr}}{r}(\hat n\times\vec m)\times\hat n$, $\vec B_{EQ} = -\frac{\mu_0 ick^3}{24\pi}\frac{e^{ikr}}{r}\hat n\times\vec Q(\hat n)$,
$E = Z_0(H\times\hat n)$, $Q_i(\hat n) = \sum_j Q_{ij}n_j$ and $Q_{ij} = \int\left(3x_i'x_j' - x'^2\delta_{ij}\right)\rho(x')d^3x'$]



a) $\vec p = 0$

$\vec m = \frac{2e}{T}\pi l^2\hat z = \frac{1}{2}e\omega l^2\hat z$ ← not time dep

$Q_{xx} = el^2[1+3\cos(\omega t')]$

$Q_{xy} = 3el^2\sin(\omega t') = Q_{yx}$

$Q_{zz} = el^2[1-3\cos(\omega t')]$

b) Electric quadrupole radiation is the lowest order radiation (m constant will not produce radiation).

Plug a_{ij} into formulas !

5. An un-polarized plane electromagnetic wave is scattered by a free electron. Derive the differential cross-section for scattering in the non-relativistic limit (Thompson scattering). (20 points)

$$\vec{E}_i = \vec{E}_0 e^{-i(\omega t - \vec{k}\cdot\vec{x})}$$

$$\vec{F} = m\ddot{\vec{x}} \cong -e\vec{E}_0 e^{-i\omega t} \quad \text{assuming } e^{-i\omega(t - \cancel{\frac{1}{c}\hat{k}\cdot})}$$

Solved by $\vec{x} = \vec{x}_0 e^{-i\omega t}$ where $\vec{x}_0 = \frac{e\vec{E}_0}{m\omega^2}$

$\Rightarrow$ induced dipole moment is

$$\vec{p} = -e\vec{x} = -\frac{e^2 \vec{E}_0}{m\omega^2} e^{-i\omega t}$$

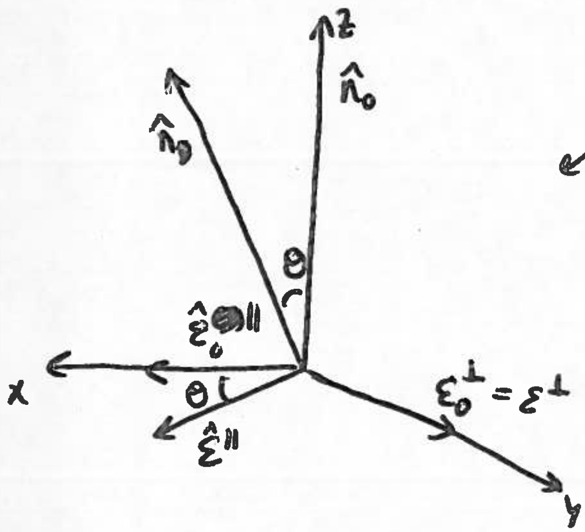
$$\Rightarrow \vec{E}_{scat} = \frac{k^2}{4\pi\varepsilon_0^2} \frac{e^{ikr}}{r} \left[(\hat{n}\times\vec{p})\times\hat{n}\right]$$

So $\frac{d\sigma}{d\Omega} = \frac{r^2 |\hat{\varepsilon}^*\cdot\vec{E}_{sc}|^2}{|\hat{\varepsilon}_0\cdot\vec{E}_{ic}|^2}$ where $\hat{\varepsilon}_0 = \frac{\vec{E}_0}{|E_0|}$

$$= \frac{\left(\frac{k^2}{4\pi\varepsilon_0^2}\right)^2 \left|\left[\cancel{(p\cdot n)(n\cdot\varepsilon^*)} - p\cdot\varepsilon^*\right]\right|^2}{|E_0|^2} = \left(\frac{k^2}{4\pi\varepsilon_0}\right)^2 \left(\frac{e^2}{m\omega^2}\right)^2 |\varepsilon^*\cdot\varepsilon_0$$

Since the incident wave was unpolarized, we need to average over $\hat{\varepsilon}_0$.

choose coordinates so that



← when $\varphi = 0$

In general,

$$\hat{\varepsilon}_0^{\parallel} \cdot \hat{\varepsilon}^{\parallel} = \cos\theta \cos\varphi$$

$$\hat{\varepsilon}_0^{\perp} \cdot \hat{\varepsilon}^{\perp} = \sin\varphi$$

average over $\varphi : 0 \to 2\pi$

$$\frac{d\sigma_{\parallel}}{d\Omega} = \left(\frac{k^2 e^2}{4\pi\varepsilon_0 m\omega^2}\right)^2 \left| \varepsilon_{\parallel}^* \cdot \varepsilon_0 \right|^2$$

$$= \left(\frac{k^2 e^2}{4\pi\varepsilon_0 m\omega^2}\right)^2 \cos^2\theta \underbrace{\langle \cos^2\varphi \rangle}_{\frac{1}{2}}$$

$$\frac{d\sigma_{\perp}}{d\Omega} = \left(\frac{k^2 e^2}{4\pi\varepsilon_0 m\omega^2}\right)^2 \left| \varepsilon_{\perp}^* \cdot \varepsilon_0 \right|^2$$

$$= \left(\frac{k^2 e^2}{4\pi\varepsilon_0 m\omega^2}\right)^2 \underbrace{\langle \sin^2\varphi \rangle}_{\frac{1}{2}}$$

$$\Rightarrow \boxed{\frac{d\sigma}{d\Omega} = \frac{d\sigma_{\parallel}}{d\Omega} + \frac{d\sigma_{\perp}}{d\Omega} = \frac{1}{2}\left(\frac{k^2 e^2}{4\pi\varepsilon_0 m\omega^2}\right)^2 (1 + \cos^2\theta)}$$