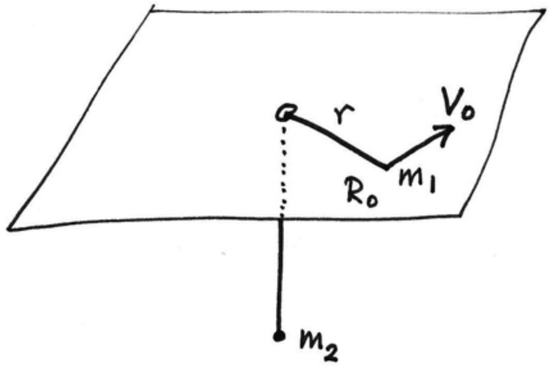
Physics & Astronomy Comprehensive Exam, UCLA, Fall 2012
(*For those taking the exam for the first time*)

1. Classical Mechanics (Version 1)

A mass m_1 moves around a hole on a frictionless horizontal plane. The mass is tied to a massless string of fixed length which passes through the hole. A mass m_2 is tied to the other end of the string and is subject to uniform gravity with acceleration constant g (see figure below).



(a) Given the initial position $\mathbf{R_0}$ and velocity $\mathbf{V_0}$ in the plane of the table and the masses m_1 and m_2, find the equation that determines the maximum and minimum radial distances of the orbit. (Do not attempt to solve this equation.)
(b) Find the frequency of oscillation of the radius of the orbit when the orbit is only slightly different from circular.

2. Classical Mechanics (Version 1)

A *K*-meson of mass m_K = 494 MeV decays into a μ-meson of mass m_μ = 106 MeV and a neutrino of approximately zero mass m_ν = 0. Calculate the kinetic energies of the μ-meson and of the neutrino for a *K*-meson decaying while at rest.

Physics & Astronomy Comprehensive Exam, UCLA, Fall 2012
(*For those repeating the exam*)

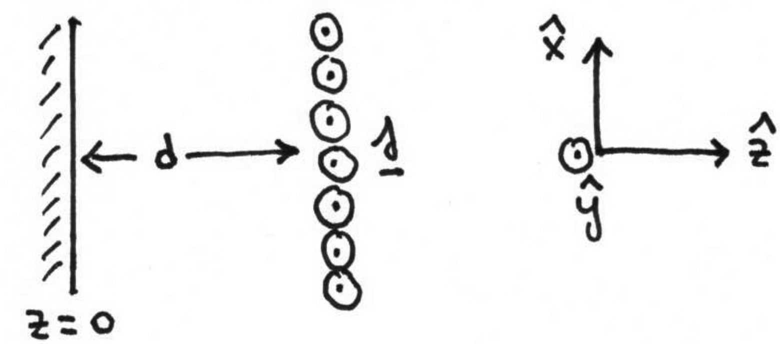
1. Statistical Mechanics (Version 2)

Calculate the lowest non-zero contribution to the specific heat C_V of an ideal Fermi gas at low temperatures T, as a function of the one-particle density of states $D(\varepsilon)$ at energy ε.
The value of the following integral may be helpful,

$$\int_{-\infty}^{\infty} dx \frac{x^2}{(e^x + e^{-x})^2} = \frac{\pi^2}{24}$$

2. *Electromagnetism* (Version 2)

A current sheet of infinite extent along the y and x-directions is at a distance d from an infinite planar wall located at $z = 0$, as sketched in the diagram below.



The current density of the sheet oscillates in time t, and is given by

$$\vec{j}(x,y,z) = \lambda\cos(\omega\, t)\delta(z-d)\hat{y}\ .$$

The wall exhibits a power reflection coefficient R at frequency ω. The remaining regions are vacuum.

(a) Apply complex notation to find the electric and magnetic field vectors for $z > 0$. Use the Cartesian coordinates indicated in the diagram.

(b) Find the time averaged power per unit area required to maintain the current flowing in the sheet.

3. Quantum Mechanics

The magnetic dipole moment of the electron is $\boldsymbol{\mu} = -2\mu_B \mathbf{S}$, where $\mu_B = \dfrac{e\hbar}{2mc}$ is the Bohr magneton ($e > 0$), **S** the spin.

(a) Write down the Hamiltonian for an electron in a uniform magnetic field **B** along the z-axis (we consider only spin degrees of freedom in this problem, i.e. no momentum).

(b) At $t = 0$, the state of the electron is: $|\psi(0)\rangle = \dfrac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ (where the notation is referred to the Pauli matrices). Find $|\psi(t)\rangle$, and $\langle \hat{S}_x \rangle$, $\langle \hat{S}_y \rangle$ on the state $|\psi(t)\rangle$. Describe your result in physical terms (i.e. what is the spin doing?).

(c) Now introduce a small uniform magnetic field $\mathbf{B}_1$ along the x-axis. Calculate in perturbation theory the first non-zero correction to the energy levels of the Hamiltonian of part (a). Show that your result is consistent with the exact result (which you can easily calculate).

4. Quantum Mechanics

Consider a particle subject to a one-dimensional simple harmonic oscillator potential, whose Hamiltonian is

$$H = \frac{p^2}{2m} + \frac{kx^2}{2},$$

where $p = -i\hbar\partial_x$ is the particle's momentum operator, m mass, and k spring constant. Suppose that at $t = 0$ the state vector is given by

$$e^{-ipa/\hbar}|0\rangle,$$

where $|0\rangle$ is the ground state and a is some number with dimension of length. Evaluate the expectation value of position $\langle x\rangle$ for $t \geq 0$.

5. Quantum Mechanics

A free, spinless non-relativistic particle with mass m and charge q is moving in a uniform magnetic field $\mathbf{B} = B\mathbf{z}$. Find the spectrum of energy eigenvalues.

(a) Write down the Hamiltonian H in terms of the kinetic momentum π, including the magnetic field via the minimal substitution $\mathbf{p}\rightarrow\pi = \mathbf{p} - q\mathbf{A}$, where $\mathbf{p}$ is the canonical momentum and $\mathbf{A}$ is the vector potential.

(b) Choose a convenient gauge and write $\mathbf{A}$ in terms of B.

(c) Calculate $[\pi_x, \pi_y]$.

(d) Write $\pi_x = i\,\alpha\,(\mathbf{a}-\mathbf{a}^\dagger)$ and $\pi_y = \alpha\,(\mathbf{a}+\mathbf{a}^\dagger)$, where α is a constant, $\mathbf{a}$ is an annihilation operator, and $\mathbf{a}^\dagger$ is a creation operator. Find α.

(e) Find the eigenvalues of H.

6. *Quantum Mechanics*

A particle in a spherically symmetrical potential is known to be in an eigenstate of $\mathbf{L}^2$ and L_z with eigenvalues $\hbar^2 l(l+1)$ and $m\hbar$, respectively, denoted by $|lm\rangle$. $\mathbf{L}$ is the angular momentum operator, whose components obey the usual commutation algebra. Prove that the expectation values involving L_x and L_y obey

$$\langle L_x \rangle = \langle L_y \rangle = 0, \qquad \langle L_x^2 \rangle = \langle L_y^2 \rangle = \frac{l(l+1)-m^2}{2}\hbar^2$$

in the eigenstate $|lm\rangle$.

7. *Quantum Mechanics*

A system that has three unperturbed states can be represented by the perturbed Hamiltonian matrix

$$\begin{pmatrix} E_1 & 0 & a \\ 0 & E_1 & b \\ a^* & b^* & E_2 \end{pmatrix}.$$

The quantities a and b are to be regarded as perturbations that are of the same order and are small compared with $E_2 - E_1$.

(a) Use the second-order nondegenerate perturbation theory to calculate the perturbed eigenvalues.
(b) Diagonalize the matrix to find the exact eigenvalues.
(c) Use the second-order degenerate perturbation theory. Compare the results obtained.

8. Statistical Mechanics

This problem concerns the fundamental definitions of thermodynamics. The numbers have been kept small to minimize the mathematics – use discrete differences in place of derivatives where appropriate.

A system has N=2 distinguishable particles, each of which can have energy $0, \varepsilon, 2\varepsilon, 3\varepsilon, \ldots\infty\varepsilon$. Say the system has total energy E=5ε.

(a) What is the entropy S?

(b) What is the temperature T?

(c) What is the chemical potential μ? Determine this by adding a 3rd particle to the system.

Imagine a different system with N=6 distinguishable particles, each of which can have energy 0 or ε.

(d) Plot the entropy as a function of the different discrete values of the total E.

(e) For what value of E is the temperature maximized?

(f) Plot T as a function of E for the case where N is large (but fixed), and comment on all of the various limiting cases.

9. Statistical Mechanics

Consider capillary (surface tension driven) waves at the free **surface** of a film of liquid He of area A. The dispersion relation is:

$$\omega = \sqrt{\frac{\sigma}{\rho}}\, k^{3/2} \qquad (\sigma \text{ is the surface tension and } \rho \text{ the density})$$

Treating these excitations as quasi-particles obeying Bose statistics, find the contribution of these capillary waves to the heat capacity of the film, specifically, the temperature dependence of this contribution. (Hint: the energy of the quasi-particle is, of course, $E_{\mathrm{k}} = \hbar\,\omega$.)

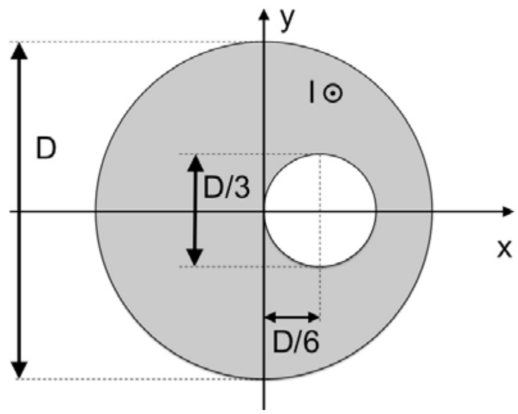
10. Statistical Mechanics

Consider a collection of N classical and spinless, non-interacting charged particles of charge q and mass m in a region of volume V in which a uniform magnetic field points along the z-direction, i.e., $\vec{B} = B_0\hat{z}$, with B_0 a constant. The charges are in good contact with a heat reservoir at temperature T.

(a) Deduce what is the equation of state for this magnetized system.
(b) Find the average induced magnetization $\langle M \rangle$ for this system.
(c) Find the relative (i.e., percentage) magnitude of the RMS fluctuation in average energy $\langle E \rangle$.

11. Electromagnetism

Consider an infinitely long cylindrical conductor of diameter D with an infinitely long cylindrical channel cut into it. The channel has a circular cross section with diameter D/3 and is offset from the axis of the conductor by D/6. The cylinder carries a current I out from the paper plane, uniformly distributed across the solid part of the conductor.



(a) Calculate the magnetic field everywhere on the X-Z plane (i.e. the axis that contains the axis of the two cylinders).
(b) Calculate the magnetic field inside the cylindrical hole.

12. Electromagnetism

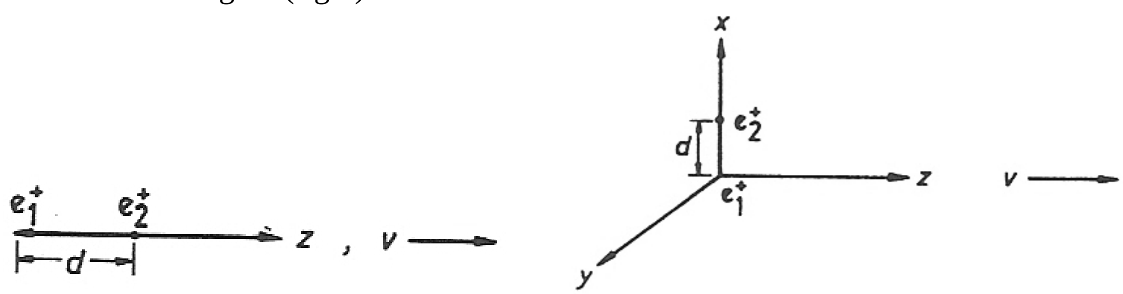
A pulsar emits bursts of radio waves which are observed from the Earth at two different frequencies, ω_1 and ω_2. An astronomer notes that the arrival time of the bursts is delayed at the lower frequency, that is, the pulse at ω_1 arrives after the pulse at ω_2. The delay, τ is due to dispersion in the interstellar medium. Assume this medium consists of ionized hydrogen (called the dilute plasma), m_e is the mass of the electron and N the number of electrons per unit volume.

(a) Find the index of refraction of the dilute plasma.

(b) Find the distance from the pulsar to the Earth.

13. Electromagnetism

(a) Consider two positrons in a beam at SLAC. The beam has energy of about 50 GeV ($\gamma \approx 10^5$). In the beam (rest) frame, they are separated by a distance *d*, and positron e_2^+ is traveling directly ahead of e_1^+ in the Z-axis, shown in the figure (left) below. Write down $\vec{E}$, $\vec{B}$, the Lorentz force $\vec{F}$ and the acceleration $\vec{a}$ on e_1^+ exerted by e_2^+. Do this in both the rest and laboratory frames.
(b) The problem is the same as in part (a) except this time the two positrons are traveling side by side as shown in the figure (right) below.



Two positrons separated by a distance of *d* travel with a velocity of *v* in the Z axis.

14. Electromagnetism

Consider a dielectric medium of infinite extent in all directions. The medium has a tensor dielectric (at zero frequency) given by

$$\vec{\vec{\varepsilon}} = \begin{pmatrix} \varepsilon_{xx} & 0 & 0 \\ 0 & \varepsilon_{yy} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{pmatrix}$$

with $\varepsilon_{xx} = \varepsilon_{yy} \equiv \varepsilon_{\perp} \neq \varepsilon_{zz}$, and where ($x$, y, z) refer to Cartesian coordinates. A point charge of charge q is placed at the origin of the coordinate system.

(a) Find the magnitude of the electric field at an arbitrary point (x, y, z), i.e., $\left|\vec{E}\right|$.

(b) Deduce the polarization charge density ρ_p induced on the dielectric at an arbitrary point (x, y, z).

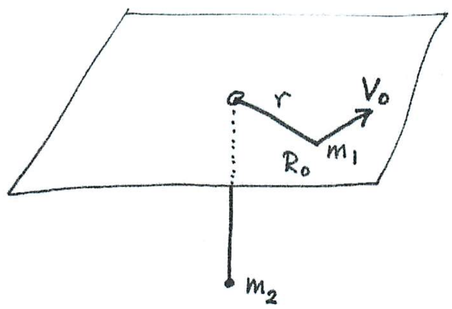
(c) Find the total electrical energy density u_E at (x, y, z).

QUESTION 1

VERSION 1

Question 1

A mass m_1 moves around a hole on a frictionless horizontal plane. The mass is tied to a massless string of fixed length which passes through the hole. A mass m_2 is tied to the other end of the string and is subject to uniform gravity with acceleration constant g (see fig 1).
(a) Given the initial position $\mathbf{R}_0$ and velocity $\mathbf{V}_0$ in the plane of the table and the masses m_1 and m_2, find the equation that determines the maximum and minimum radial distances of the orbit. (Do not attempt to solve this equation.)
(b) Find the frequency of oscillation of the radius of the orbit when the orbit is only slightly different from circular.



Solution to Question 1

The system is conservative and subject to holonomic constraints; thus we use the Lagrangian formulation of mechanics to derive the equations of motion. We use polar coordinates (r, θ) in the horizontal plane to parametrize the position of m_1 (and thus of m_2).

(a) The Lagrangian L for the combined system is given by,

$$L = \frac{1}{2}m_1(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2}m_2\dot{r}^2 - m_2 g r \tag{0.1}$$

The canonical momenta are,

$$\begin{aligned} p_r &= (m_1 + m_2)\dot{r} \\ p_\theta &= m_1 r^2 \dot{\theta} \end{aligned} \tag{0.2}$$

Since L is independent of θ, the angular momentum p_θ is conserved during the motion. The Euler-Lagrange equation for r is given by,

$$(m_1 + m_2)\ddot{r} - m_1 r\dot{\theta}^2 + m_2 g = 0 \tag{0.3}$$

Eliminating $\dot{\theta}$, we find a reduced equation for r only,

$$(m_1 + m_2)\ddot{r} - \frac{p_\theta^2}{m_1 r^3} + m_2 g = 0 \tag{0.4}$$

By multiplying through by $\dot{r}$, we integrate the resulting equation to get the total energy ε,

$$\frac{1}{2}(m_1 + m_2)\dot{r}^2 + \frac{p_\theta^2}{2m_1 r^2} + m_2 g r = \varepsilon \tag{0.5}$$

The constants of motion p_θ and ε are determined by the initial conditions, which we cast in the following form,

$$\begin{aligned} r = R_0 \qquad\qquad & \dot{r} = V_0 \cos\phi \\ & r\dot{\theta} = V_0 \sin\phi \end{aligned} \tag{0.6}$$

so that

$$\begin{aligned} p_\theta &= m_1 R_0 V_0 \sin\phi \\ \varepsilon &= \frac{1}{2} m_1 V_0^2 + \frac{1}{2} m_2 V_0^2 \cos^2\phi + m_2 g R_0 \end{aligned} \tag{0.7}$$

The condition for extremal radius $r = r_e$ is given by setting $\dot{r} = 0$, and we find,

$$\frac{p_\theta^2}{2m_1 r_e^2} + m_2 g r_e = \varepsilon \tag{0.8}$$

This equation is equivalent to a 3-rd order polynomial equation which always has one physically unacceptable negative root, and two positive roots representing the minimum and the maximum radius.

(b) Circular motion corresponds to $r = R_0$ and $\phi = \pi/2$. Small deviations from circular motion may be parametrized by $r = R_0 + x$ with $|x| \ll R_0$, and $\phi - \pi/2$ small. Since the corrections to p_θ due to the perturbation in ϕ are second order, we may neglect those. Thus, the fluctuation equation becomes,

$$(m_1 + m_2)\ddot{x} + \frac{3p_\theta^2}{m_1 R_0^4} x = 0 \tag{0.9}$$

Using the equilibrium condition for circular motion, $p_\theta^2 = m_1 m_2 R_0^3 g$, we find the following frequency ω of small oscillations,

$$\omega^2 = \frac{3m_2 g}{(m_1 + m_2)R_0} \tag{0.10}$$

Q2, V1

Question 2

A K-meson of mass $m_K = 494$ MeV decays into a μ-meson of mass $m_\mu = 106$ MeV and a neutrino of approximately zero mass $m_\nu = 0$. Calculate the kinetic energies of the μ-meson and of the neutrino for a K-meson decaying while at rest.

Solution to Question 2

Conservation of momentum implies that the momentum $\mathbf{p}_\mu$ of the μ-meson and the momentum $\mathbf{p}_\nu$ of the neutrino are opposite to one another $\mathbf{p}_\mu = -\mathbf{p}_\nu$. Thus the energy of the neutrino is given by $E_\nu = |\mathbf{p}_\nu|c = |\mathbf{p}_\mu|c$. Conservation of energy reads,

$$m_K c^2 = |\mathbf{p}_\mu|c + \sqrt{\mathbf{p}_\mu^2 c^2 + m_\mu^2 c^4} \tag{0.11}$$

Solving for $|\mathbf{p}_\mu|$, we find,

$$|\mathbf{p}_\mu| = \frac{m_K^2 - m_\mu^2}{2m_K}\, c \tag{0.12}$$

The kinetic energy T_ν of the neutrino and of the kinetic energy T_μ of the μ-meson are,

$$\begin{aligned} T_\nu &= \frac{m_K^2 - m_\mu^2}{2m_K}\, c^2 \\ T_\mu &= \frac{m_K^2 + m_\mu^2}{2m_K}\, c^2 - m_\mu c^2 \end{aligned} \tag{0.13}$$

Numerical evaluation gives approximate values,

$$T_\nu = 236\,\text{MeV} \qquad\qquad T_\mu = 152\,\text{MeV} \tag{0.14}$$

Question 1, VERSION 2

Calculate the lowest non-zero contribution to the specific heat C_V of an ideal Fermi gas at low temperatures T, as a function of the one-particle density of states $D(\varepsilon)$ at energy ε. The value of the following integral may be helpful,

$$\int_{-\infty}^{\infty} dx \, \frac{x^2}{(e^x + e^{-x})^2} = \frac{\pi^2}{24} \tag{0.15}$$

Solution to Question 3

In terms of the density of one-particle states $D(\varepsilon)$ and the Fermi occupation number,

$$f(\varepsilon) = \frac{1}{e^{\beta(\varepsilon-\mu)} + 1} \qquad\qquad \beta = \frac{1}{kT} \tag{0.16}$$

the total number of particles N and the internal energy E are given by,

$$\begin{aligned} N &= \int_0^\infty d\varepsilon \, D(\varepsilon) f(\varepsilon) \\ E &= \int_0^\infty d\varepsilon \, \varepsilon D(\varepsilon) f(\varepsilon) \end{aligned} \tag{0.17}$$

(as usual, we have assumed that $D(\varepsilon) = 0$ for $\varepsilon < 0$). We also define the chemical potential μ_0 at zero temperature by the relation,

$$N = \int_0^{\mu_0} d\varepsilon \, D(\varepsilon) \tag{0.18}$$

Since the function $D(\varepsilon)$ does not involve temperature, the specific heat is given by,

$$C_V = \frac{\partial E}{\partial T} = \int_0^\infty d\varepsilon \, \varepsilon D(\varepsilon) \frac{\partial f(\varepsilon)}{\partial T} \tag{0.19}$$

Working this out, and after some minor simplifications, we get,

$$C_V = \frac{1}{kT^2} \int_0^\infty d\varepsilon \, \frac{\varepsilon(\varepsilon - \mu) D(\varepsilon)}{\left(e^{\beta(\varepsilon-\mu)/2} + e^{-\beta(\varepsilon-\mu)/2}\right)^2} \tag{0.20}$$

For low temperatures, the denominator is responsible for concentrating the support of the integral over ε near μ, so we may extend the integration region all the way to $-\infty$. Also, to leading order, we may evaluate $D(\varepsilon)$ at the central value μ. Finally, the parity of the remaining integral allows us to replace the factor ε in the numerator by $\varepsilon - \mu$, so that we end up with the following expression,

$$C_V = \frac{D(\mu)}{kT^2} \int_{-\infty}^{\infty} d\varepsilon \, \frac{(\varepsilon - \mu)^2}{\left(e^{\beta(\varepsilon-\mu)/2} + e^{-\beta(\varepsilon-\mu)/2}\right)^2} \tag{0.21}$$

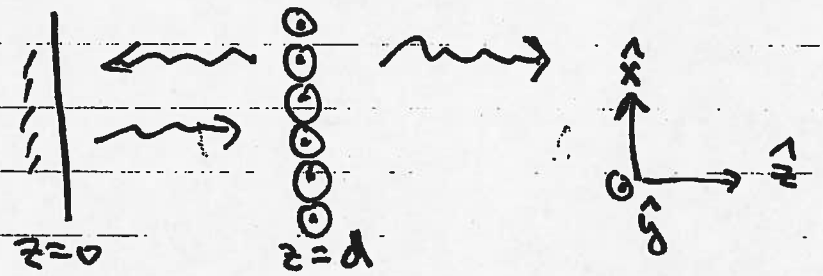
Changing variables from ε to x with $\varepsilon = \mu + 2kTx$ gives,

$$C_V = 8k^2 T D(\mu) \int_{-\infty}^{\infty} dx \, \frac{x^2}{(e^x + e^{-x})^2} \tag{0.22}$$

Finally, using the value of the integral stated in the problem, and using the fact that for small temperatures we have $\mu = \mu_0 + \mathcal{O}(T^2)$, we may approximate this result further by setting $D(\mu) = D(\mu_0)$, so that the final result is given by,

$$C_V = \frac{1}{3}\pi^2 k^2 D(\mu_0)\, T \tag{0.23}$$

E+M # 2 (answer)



a) using complex notation $\underline{J} = \lambda\, \text{Re}\, e^{-i\omega t}\, \delta(z-d)\, \hat{y}$

For $z > d$: $\underline{E} = \text{Re}\left[\tilde{E}_+ e^{i[k(z-d)-\omega t]}\right](-\hat{y})$

$$\underline{B} = \text{Re}\left[\tilde{E}_+ e^{i[k(z-d)-\omega t]}\right](+\hat{x})$$

For $0 < z < d$:

$$\underline{E} = \text{Re}\left[\tilde{E}_- e^{-i[k(z-d)+\omega t]}\right](-\hat{y})$$
$$+\text{Re}\left[\sqrt{R}\,\tilde{E}_- e^{ikd}\, e^{i[kz-\omega t]}\right](+\hat{y})$$

reflect

$$\underline{B} = \text{Re}\left[\tilde{E}_- e^{-i[k(z-d)+\omega t]}\right](-\hat{x})$$
$$+\text{Re}\left[\sqrt{R}\,\tilde{E}_- e^{ikd}\, e^{i[kz-\omega t]}\right](-\hat{x})$$

R is power $\Rightarrow \sqrt{R}$ is amplitude reflection coefficient

b) Power per unit volume is $\frac{dP}{dV} = \underline{E} \cdot \underline{J}$

integrate over sheet and time average

$$\left\langle \frac{dP}{dA} \right\rangle = \left\langle \int_{-\infty}^{\infty} dz \; \underline{E} \cdot \frac{\lambda}{2} \left(e^{-i\omega t} + e^{+i\omega t} \right) \delta(z-d) \, \hat{y} \right\rangle$$

$$\boxed{\left\langle \frac{dP}{dA} \right\rangle = -\frac{1}{2} \lambda \, \mathrm{Re}\left(\tilde{E}_+ \right)}$$ ⇒ need to determine $\tilde{E}_+$ from Boundary conditions

For electric field: $\boxed{\Delta E_y = 0}$ for all t at $z = d$

$$\Rightarrow \quad -\tilde{E}_+ = -\tilde{E}_- + \sqrt{R}\, E_- e^{i2kd}$$

$$\Rightarrow \quad \tilde{E}_+ = \left(1 - \sqrt{R}\, e^{i2kd}\right) \tilde{E}_-$$

For magnetic field: $\boxed{\Delta B_x = \frac{4\pi}{c} \lambda}$ for all t at $z = d$

$$\tilde{E}_+ - \left[-\tilde{E}_- - \sqrt{R}\, e^{i2kd} \tilde{E}_- \right] = \frac{4\pi}{c} \lambda$$

$$\tilde{E}_+ + \left[1 + \sqrt{R}\, e^{i2kd} \right] \tilde{E}_- = \frac{4\pi}{c} \lambda$$

but

$$\left[1 - \sqrt{R}\, e^{i2kd} \right] \tilde{E}_- + \left[1 + \sqrt{R}\, e^{i2kd} \right] \tilde{E}_- = \frac{4\pi}{c} \lambda$$

$$\Rightarrow \quad \tilde{E}_- = \left(\frac{1}{2}\right)\left(\frac{4\pi}{c}\lambda\right)$$

$$\Rightarrow \quad \boxed{E_+ = \frac{2\pi}{c} \lambda \left(1 - \sqrt{R}\, e^{i2kd} \right)}$$

and the power needed to maintain current is:

$$\boxed{\left\langle \frac{dP}{dA} \right\rangle = -\frac{\pi \lambda^2}{c} \left(1 - \sqrt{R} \cos 2kd \right)}$$

Q3

4) Electron spin in magn. field:

$$\hat{H} = -\vec{\mu} \cdot \vec{B} = 2\mu_B B \hat{S}_z = \mu_B B \hat{\sigma}_z$$

$$\Rightarrow \quad \hat{H} = \mu_B B \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$t = 0$: $\quad |\psi(0)\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

written in terms of eigenvectors of $\hat{H}$ (eigenvalues are $\pm \mu_B B$)

$$\Rightarrow \quad |\psi(t)\rangle = \frac{1}{\sqrt{2}} e^{i\mu_B B t/\hbar} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2}} e^{-i\mu_B B t/\hbar} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

To calculate $\langle S_x \rangle$, note that $\hat{\sigma}_x \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$,

$\hat{\sigma}_x \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \Rightarrow \langle \hat{\sigma}_x \rangle = \langle \psi(t) | \hat{\sigma}_x | \psi(t) \rangle$

$$= \frac{1}{2} e^{2i\mu_B B t/\hbar} + \frac{1}{2} e^{-2i\mu_B B t/\hbar}$$

$$\Rightarrow \quad \langle \hat{\sigma}_x \rangle = \cos\left(\frac{2\mu_B B}{\hbar} t\right) = \cos(\omega t)$$

$$\boxed{\omega = \frac{2\mu_B B}{\hbar}} \quad .$$

Similarly, $\hat{\sigma}_y \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ i \end{pmatrix}$, $\hat{\sigma}_y \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -i \\ 0 \end{pmatrix}$

and $\langle \hat{\sigma}_y \rangle = \frac{1}{2} i e^{2i\mu_B B t/\hbar} - \frac{1}{2} i e^{-2i \cdots}$

$$= -\sin(\omega t)$$

$\rightarrow$ the spin precesses with ang.-freq. ω around the direction of $\vec{B}$:

(clockwise looking down on $\vec{B}$, which is right since $q_e < 0$).

c) $\hat{V} = 2\mu_B B_1 \hat{S}_x$

$$= \mu_B B_1 \hat{\sigma}_x = \mu_B B_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{perturbation}$$

$E^{(1)} = \mu_B B_1 \, (1, 0) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$ for the eigenv. $E^{(0)} = \mu_B B$ and same for the eigenv. $E^{(0)} = -\mu_B B$

i.e. first order corrections are zero.

$$E^{(2)} = \pm \frac{1}{2\mu_B B} |V_{12}|^2$$ are the second order corrections for $E^{(0)} = \pm \mu_B B$

$$V_{12} = \mu_B B_1 (1, 0) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \mu_B B_1$$

$$\Rightarrow \quad E^{(2)} = \pm \frac{(\mu_B B_1)^2}{2\mu_B B} = \pm \frac{1}{2} \mu_B B \left(\frac{B_1}{B}\right)^2 .$$

The exact result is of course $E = \pm \mu_B B_{tot}$

where $B_{tot} = \sqrt{B^2 + B_1^2}$

and since $B_{tot} = B^2 \sqrt{1 + (B_1/B)^2} \simeq B^2 \left[1 + \frac{1}{2}\left(\frac{B_1}{B}\right)^2\right]$

to lowest order you got $E^{(1)} = \pm \mu_B B \frac{1}{2} \left(\frac{B_1}{B}\right)^2 .$

2012 Comprehensive Exam

Q.4 *QM1*

Consider a particle subject to a one-dimensional simple harmonic oscillator potential, whose Hamiltonian is

$$H = \frac{p^2}{2m} + \frac{kx^2}{2},$$

where $p = -i\hbar\partial_x$ is the particle's momentum operator, m mass, and k spring constant. Suppose that at $t = 0$ the state vector is given by

$$e^{-ipa/\hbar}|0\rangle,$$

where $|0\rangle$ is the ground state and a is some number with dimension of length. Evaluate the expectation value of position $\langle x \rangle$ for $t \geq 0$.

Solution: At $t = 0$,

$$\langle p \rangle_0 = \langle 0|e^{ipa/\hbar} p e^{-ipa/\hbar}|0\rangle = \langle 0|p|0\rangle = 0$$

and

$$\langle x \rangle_0 = \langle 0|e^{ipa/\hbar} x e^{-ipa/\hbar}|0\rangle = \langle 0|e^{ipa/\hbar}[x, e^{-ipa/\hbar}]|0\rangle = i\hbar\langle 0|e^{ipa/\hbar}\partial_p e^{-ipa/\hbar}|0\rangle = a\,.$$

Using harmonic oscillator equations of motion,

$$\dot{x} = \frac{i}{\hbar}[H, x] = \partial_p H = \frac{p}{m} \quad \text{and} \quad \dot{p} = \frac{i}{\hbar}[H, p] = -\partial_x H = -kx\,,$$

we immediately find the oscillatory solution in the form

$$\langle x \rangle_t = a\cos(\omega t)\,,$$

were $\omega = \sqrt{k/m}$ is the oscillator natural frequency.

Q5

Quantum mechanics

1) A free, spinless non-relativistic particle with mass m and charge q is moving in a uniform magnetic field **B**=B**z**. Find the spectrum of energy eigenvalues.

a) Write down the Hamiltonian H in terms of the conjugate momentum $\boldsymbol{\pi}$, including the magnetic field via the minimal substitution $\mathbf{p}\rightarrow\boldsymbol{\pi}=\mathbf{p}-q\mathbf{A}$.
b) Choose a convenient gauge and write the vector potential **A** in terms of B.
c) Calculate $[\pi_x, \pi_y]$.
d) Write $\pi_x = i\,c\,(\mathbf{a}-\mathbf{a}^\dagger)$ and $\pi_y = c\,(\mathbf{a}+\mathbf{a}^\dagger)$, where c is a constant, **a** is an annihilation operator, and $\mathbf{a}^\dagger$ is a creation operator. Find c.
e) Find the eigenvalues of H.

a) $H = \frac{\Pi^2}{2m} = \frac{(\vec{p} - q\vec{A})^2}{2m}$

b) We require $\vec{A}$ such that $\nabla \times A = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = B\hat{k}$

$\Rightarrow$ Choose $A_y = xB$

$\vec{A} = (0, xB, 0)$

c) $[\Pi_x, \Pi_y] = [p_x - qA_x, p_y - qA_y] = [p_x, p_y - qxB]$

$= -qB[p_x, x] = i\hbar qB$

d) $[a, a^\dagger] = 1$ $\quad [\Pi_x, \Pi_y] = ic^2[a - a^\dagger, a + a^\dagger]$

$= ic^2([a, a^\dagger] - [a^\dagger, a]) = 2ic^2 = i\hbar qB$

$c = \pm\sqrt{\frac{\hbar qB}{2}}$

e) $H = \frac{\Pi_x^2 + \Pi_y^2 + \Pi_z^2}{2m} = \frac{1}{2m}\left(-\frac{\hbar qB}{2}(a - a^\dagger)(a - a^\dagger) + \frac{\hbar qB}{2}(a + a^\dagger)(a + a^\dagger) + p_z^2\right)$

$= \frac{1}{2m}\left(\frac{\hbar qB}{2}\right)\left(\cancel{-aa} - \cancel{a^\dagger a^\dagger} + aa^\dagger + a^\dagger a + \cancel{aa} + \cancel{a^\dagger a^\dagger} + a^\dagger a + aa^\dagger\right) + \frac{p_z^2}{2m}$

$= \frac{\hbar qB}{4m}(4a^\dagger a + 2) + \frac{p_z^2}{2m} = \hbar\frac{qB}{m}\left(n + \frac{1}{2}\right) + \frac{p_z^2}{2m}$

$E = \hbar\omega\left(n + \frac{1}{2}\right) + p_z^2/2m$ where ω is the cyclotron frequency

Q. 6

A particle in a spherically symmetrical potential is known to be in an eigenstate of $\mathbf{L}^2$ and L_z with eigenvalues $\hbar^2 l(l+1)$ and $m\hbar$, respectively, denoted by $|lm\rangle$. $\mathbf{L}$ is the angular momentum operator, whose components obey the usual commutation algebra. Prove that the expectation values involving L_x and L_y obey

$$\langle L_x \rangle = \langle L_y \rangle = 0\,, \quad \langle L_x^2 \rangle = \langle L_y^2 \rangle = \frac{l(l+1)-m^2}{2}\hbar^2$$

in the eigenstate $|lm\rangle$.

Solution: Using $[L_i, L_j] = i\hbar\epsilon_{ijk}L_k$, we evaluate (henceforth setting $\hbar = 1$)

$$\begin{aligned}
\langle L_x^2 - L_y^2 \rangle =& \langle [L_x, L_z]^2 - [L_y, L_z]^2 \rangle = 2m\langle L_x L_z L_x - L_y L_z L_y \rangle - m^2 \langle L_x^2 - L_y^2 \rangle - \langle L_x L_z^2 L_x - L_y L_z^2 L_y \rangle \\
=& 2m\langle L_x L_z L_x - L_y L_z L_y \rangle - m^2 \langle L_x^2 - L_y^2 \rangle \\
&- \langle L_x L_z [L_z, L_x] + m L_x L_z L_x - [L_y, L_z] L_z L_y - m L_y L_z L_y \rangle \\
=& m\langle L_x L_z L_x - L_y L_z L_y \rangle - m^2 \langle L_x^2 - L_y^2 \rangle \\
=& m\langle L_x [L_z, L_x] + m L_x^2 - [L_y, L_z] L_y - m L_y^2 \rangle - m^2 \langle L_x^2 - L_y^2 \rangle = 0\,.
\end{aligned}$$

Thus (restoring $\hbar$)

$$\langle L_x^2 \rangle = \langle L_y^2 \rangle = \frac{\langle \mathbf{L}^2 - L_z^2 \rangle}{2} = \frac{l(l+1)-m^2}{2}\hbar^2\,.$$

The other identities are also easily obtained:

$$i\hbar\langle L_x \rangle = \langle [L_y, L_z] \rangle = m\langle L_y - L_y \rangle = 0\,,$$

and similarly for $\langle L_y \rangle$.

$$\mathcal{H} = \begin{pmatrix} E_1 & 0 & a \\ 0 & E_1 & b \\ a^* & b^* & E_2 \end{pmatrix}.$$

Let $\mathcal{H} = \mathcal{H}_0 + V.$ where

$$V = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ a^* & b^* & 0 \end{pmatrix}$$

(a.). Using non-degenerate perturbation theory, if $E_m = E_m^0 + \Delta_m^1 + \Delta_m^2 + \cdots$

$$\Delta_m^1 = \langle m_0 | V | m_0 \rangle$$

For $|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\Delta_m^1 = V_{11} = 0.$

$|2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\Delta_m^2 = V_{22} = 0.$

$|3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $\Delta_m^3 = V_{33} = 0.$

At second - order

$$\Delta_m^{2} = \frac{\langle m_0 | V | n_0 \rangle \langle n_0 | V | m_0 \rangle}{E_0^m - E_0^n}.$$

$$\Delta_1^{(2)} = \frac{|\langle 3 | V | 1 \rangle|^2}{E_1 - E_2} = \frac{|a|^2}{E_1 - E_2}$$

$$\Delta_2^{(2)} = \frac{|\langle 3 | V | 2 \rangle|^2}{E_1 - E_2} = \frac{|b|^2}{E_1 - E_2}$$

$$\Delta_3^{(2)} = \frac{|\langle 1 | V | 3 \rangle|^2}{E_2 - E_1} + \frac{|\langle 2 | V | 3 \rangle|^2}{E_2 - E_1}$$

$$= - \frac{(|a|^2 + |b|^2)}{E_1 - E_2}.$$

(b). $\det (H - \lambda I)$.

$$= \begin{pmatrix} E_1 - \lambda & 0 & a \\ 0 & E_1 - \lambda & b \\ a^* & b^* & E_2 - \lambda \end{pmatrix}$$

$$= (E_1 - \lambda) \cdot \left\{ (E_1 - \lambda)(E_2 - \lambda) - |b|^2 \right\}$$

$$- |a|^2 \cdot (E_1 - \lambda)$$

$$= (E_1 - \lambda) \left\{ \lambda^2 - (E_1 + E_2)\lambda + E_1 E_2 - |b|^2 - |a|^2 \right\}.$$

$$\det(H - \lambda I) = 0$$

$$\Rightarrow \quad \lambda = E_1 \quad \text{or}$$

$$\lambda = \frac{E_1 + E_2 \pm \sqrt{(E_1+E_2)^2 - 4.(E_1E_2 - |b|^2 - |a|^2)}}{2}.$$

$$= \frac{E_1+E_2}{2} \pm \sqrt{\left(\frac{E_1+E_2}{2}\right)^2 - E_1E_2 + |a|^2 + |b|^2}.$$

$$= \frac{E_1+E_2}{2} \pm \sqrt{\left(\frac{E_1-E_2}{2}\right)^2 + |a|^2 + |b|^2}$$

(c) Let $|1^0\rangle$, $|2^0\rangle$ be the appropriate basis for degenerate perturbation theory. for the degenerate subspace of H_0.

Then, for $i = 1, 2$

$$(H_0 - E_1).\{|i^0\rangle + |i^1\rangle + \cdots\}$$

$$= (\Delta_i^1 + \Delta_i^2 \cdots - V).\{|i^0\rangle + |i^1\rangle + \cdots\}. \quad \text{— (A)}$$

Operate on L.H.S & R.H.S. with $\langle j^0|$. where $j \in \{1,2\}$.

We find.

$$L.H.S = 0 = (\Delta_i^1 \langle j^0|i^0\rangle - \langle j^0|V|i^0\rangle.)$$

$$+ (\Delta_i^2 \langle j^0|i^0\rangle - \langle j^0|V|i^1\rangle)$$

$$+ \cdots \quad \text{— (B)}$$

where we have grouped terms of the same order together.

Operate on (A) with $\langle 3^0| = \langle 3|$, we find

$$(E_2 - E_1)\{\langle 3^0|i^0\rangle + \langle 3^0|i'\rangle + \cdots\}$$

$$= \Delta_i^1 \langle 3^0|i^0\rangle - \langle 3^0|V|i^0\rangle$$

+ higher order terms. — (C)

From (C), we see comparing 1st order terms,

$$\langle 3^0|i'\rangle = \frac{\langle 3^0|V|i^0\rangle}{E_1 - E_2}$$

From (B), 1st order terms give us:

$\langle j^0|V|i^0\rangle = 0$, but this is true for any states in the degenerate space spanned by $|1\rangle$ & $|2\rangle$.

At 2^{nd} order, we get

$$\Delta^2 \delta_{j_0, i_0} = \langle j^0 | V | i' \rangle$$

In other words for the right basis

$$\frac{\langle j^0 | V | 3^0 \rangle \langle 3^0 | V | i^0 \rangle}{E_1 - E_2} = \Delta^2 \delta_{i_0, j_0}.$$

Thus we have to diagonalize the

matrix, $M = \dfrac{P_D V |3^0\rangle\langle 3^0| V P_D}{E_1 - E_2}$. where

$$P_D = |1\rangle\langle 1| + |2\rangle\langle 2|.$$

In the basis $, |1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$

$$M_{11} = \frac{V_{13} V_{31}}{E_1 - E_2} = \frac{|a|^2}{E_1 - E_2},$$

$$M_{12} = \frac{V_{13} V_{32}}{E_1 - E_2} = \frac{a b^*}{E_1 - E_2}.$$

$$M_{21} = \frac{a^* b}{E_1 - E_2}, \quad M_{22} = \frac{|b|^2}{E_1 - E_2}.$$

To diagonalize M,

$$\det(M - \lambda I) = 0.$$

$$\Leftrightarrow \det\begin{pmatrix} |a|^2 - \lambda & ab^* \\ a^* b & |b|^2 - \lambda \end{pmatrix} = 0.$$

$$\Leftrightarrow (|a|^2 - \lambda)(|b|^2 - \lambda) - |a|^2 |b|^2 = 0.$$

$$\Leftrightarrow \lambda^2 - \lambda(|a|^2 + |b|^2) = 0$$

$$\Leftrightarrow \lambda = 0 \quad \text{or} \quad \lambda = |a|^2 + |b|^2.$$

Thus the eigenvalues of M are

$$0, \quad \frac{|a|^2 + |b|^2}{E_1 - E_2}$$

For $|3\rangle$,

$$\Delta_3^{(2)} = \frac{|\langle 1^0 | V | 3^0 \rangle|^2 + |\langle 2^0 | V | 3^0 \rangle|^2}{E_2 - E_1}.$$

which gives the same result as non-degenerate perturbation theory,

$$\Delta_3^{(2)} = -\frac{(|a|^2 + |b|^2)}{E_1 - E_2}.$$

From the exact results,

$$E = E_1$$

or
$$E = \frac{E_1 + E_2}{2} \pm \sqrt{\frac{(E_1 - E_2)^2}{2} + |a|^2 + |b|^2}$$

$$= \frac{E_1 + E_2}{2} \pm \frac{E_1 - E_2}{2} \left(1 + \frac{1}{2} \frac{(|a|^2 + |b|^2) \cdot 2}{(E_1 - E_2)^2} + \cdots \right.$$

$$= E_1 + \frac{(|a|^2 + |b|^2)}{E_1 - E_2} + \cdots$$

or

$$E_2 - \frac{(|a|^2 + |b|^2)}{E_1 - E_2}$$

which agrees with the results of degenerate perturbation theory, but not with those of non-degenerate perturbation theory, as expected.

Q8

Statistical mechanics

2) This problem concerns the fundamental definitions of thermodynamics. The numbers have been kept small to minimize the mathematics – use discrete differences in place of derivatives where appropriate.

A system has N=2 distinguishable particles, each of which can have energy 0, ε, 2ε, 3ε, …∞ε. Say the system has total energy E=5ε.
a) What is the entropy S?
b) What is the temperature T?
c) What is the chemical potential μ? (Hint: add a 3rd particle.)

Imagine a different system with N=6 distinguishable particles, each of which can have energy 0 or ε.
d) Plot the entropy as a function of the different discrete values of the total E.
e) For what value of E is the temperature maximized?
f) Plot T as a function of E for the case where N is large (but fixed), and comment on the various limiting cases.

ε_1	ε_2
0	5ε
1	4ε
2	3ε
3	2ε
4	1ε
5	0ε

There are 6 accessible states Ω

a) $\boxed{S = k_B \ln 6}$

$$\frac{1}{T} = \frac{\partial S}{\partial E} \simeq \frac{\Delta S}{\Delta E} = \frac{S_{6\varepsilon} - S_{5\varepsilon}}{6\varepsilon - 5\varepsilon} = \frac{k_B \ln(\frac{6}{5})}{\varepsilon}$$

b) $\boxed{T = \frac{\varepsilon}{k_B \ln(\frac{6}{5})}}$

$$dE = TdS - pdV + \mu dN$$

$$\mu = \left.\frac{dE}{\partial N}\right|_S \simeq \left.\frac{\Delta E}{\Delta N}\right|_S$$

We require the entropy to remain constant as we add the 3rd particle.

ε_1	ε_2	ε_3
2	0	0
0	2	0
0	0	2
1	1	0
1	0	1
0	1	1

A total energy of 2ε gives 6 accessible states, as before.

$$\mu = \frac{2\varepsilon - 5\varepsilon}{3_{\text{particles}} - 2_{\text{particles}}} = \boxed{-3\varepsilon = \mu}$$ c)

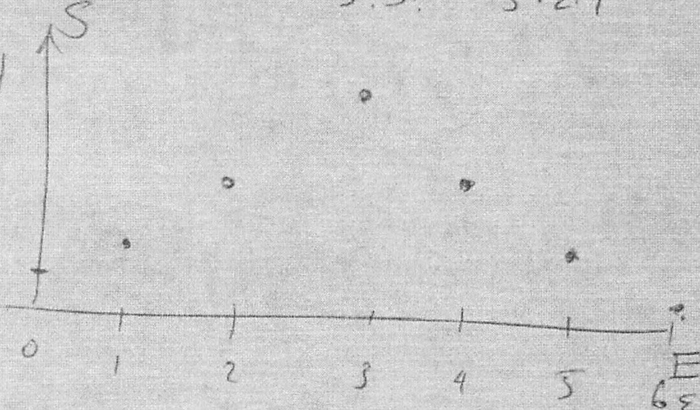
d) $N=6$, each with $\varepsilon = 0$ or ε

total energy	Ω
0	1
1	6
2	15
3	20
4	15
5	6
6	1

Ω given by binomial coefficients $\binom{N}{\frac{E}{\varepsilon}} = \binom{N}{p} = \frac{N!}{p!(N-p)!}$

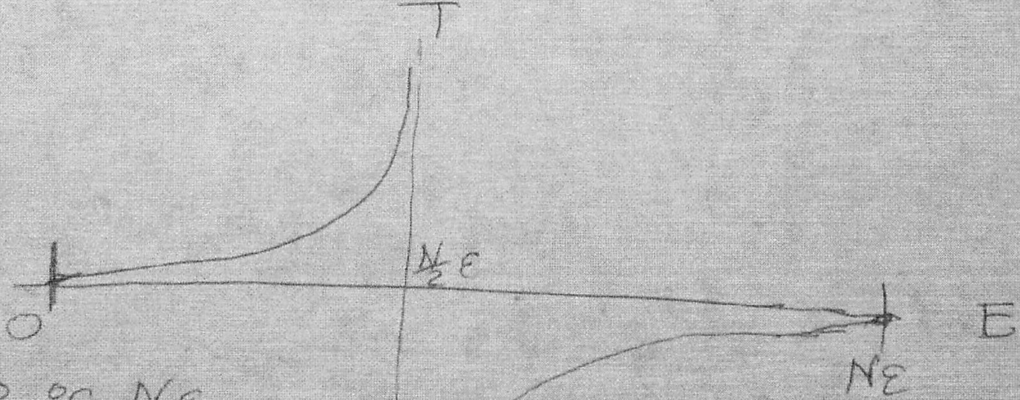
$$\frac{6!}{4!\,2!} = \frac{6\cdot 5}{2\cdot 1} = 15$$

$$\frac{6!}{3!\,3!} = \frac{6\cdot 5\cdot 4}{3\cdot 2\cdot 1} = 20$$



e) The temperature is maximized for $E = 2\varepsilon$.
($T = \frac{\partial S}{\partial E}$ is undefined at $E = 3\varepsilon$)

f)



For $E \to 0$ or $N\varepsilon$ the temperature $\to 0$ from positive or negative values respectively.

For $E \to \frac{N}{2}\varepsilon$, $T \to \pm\infty$, depending on whether the approach is from above or below. For $E > \frac{N}{2}\varepsilon$ the temperatures are negative.

Q9

Capillary waves

$\omega = \alpha k^{3/2}$, $\alpha = \sqrt{\frac{\sigma}{\rho}}$ disp. rel.

Bosons , $\mu = 0$ (number of particles not conserved)

$\rightarrow$ occupation numbers $n_k = \frac{1}{e^{E_k/T} - 1}$

$E_k = \hbar\omega$; in 2D :

$$E = \int \frac{d^2p\, d^2x}{h^2} \frac{\hbar\omega}{e^{\hbar\omega/T} - 1} \qquad \text{where } p = \hbar k$$

$$\Rightarrow E = \frac{A}{h^2} \int_0^\infty 2\pi p\, dp \frac{\hbar\omega}{e^{\hbar\omega/T} - 1} \qquad A \text{ is the area.}$$

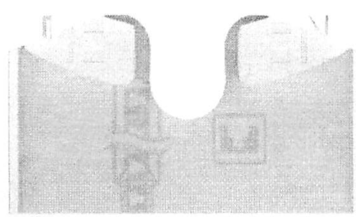
Change variables to ω : $p = \hbar k$,

$d\omega = \frac{3}{2}\alpha k^{1/2} dk$, $k^{1/2} = \left(\frac{\omega}{\alpha}\right)^{1/3}$

$$\Rightarrow E = \frac{2\pi A}{h^2} \hbar^2 \int_0^\infty d\omega \frac{2}{3\alpha}\left(\frac{\omega}{\alpha}\right)^{1/3} \frac{\hbar\omega}{e^{\hbar\omega/T} - 1}$$

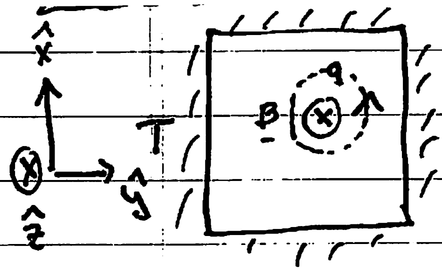
$$\frac{\hbar\omega}{T} = x \qquad = \frac{A}{3\pi(\hbar\alpha)^{4/3}} T^{7/3} \int_0^\infty dx \frac{x^{4/3}}{e^x - 1}$$

So $E \propto T^{7/3}$ and $C_A = \left.\frac{\partial E}{\partial T}\right|_A \propto T^{4/3}$.



Q 10.

SM #1 (answer)



$\underline{B} = B_0 \hat{z} \Rightarrow \underline{A} = \frac{B_0}{2}\left(-y\hat{x} + x\hat{y}\right)$

where $\underline{B} = \nabla \times \underline{A}$

Hamiltonian for magnetized particle:

$$H = \frac{1}{2m}\left(\underline{p} - \frac{q}{c}\underline{A}\right)^2 = \frac{1}{2m}\left[\left(p_x - \frac{qyB_0}{2c}\right)^2 + \left(p_y + \frac{qB_0 x}{2c}\right)^2 + p_z^2\right]$$

N-Classical, non-interacting particles have a partition function: $Z_N = \frac{1}{N!}\left[\int \frac{dp_x\, dp_y\, dp_z\, dx\, dy\, dz}{h^3} e^{-\beta H}\right]^N$

with $\beta \equiv \frac{1}{kT}$; $k \equiv$ Boltzmann constant ; $h \equiv$ Planck's const

consider:

$$\int_{-\infty}^{\infty} dp_x \int_{-\infty}^{\infty} dp_y \int_{-\infty}^{\infty} dp_z \int dx\, dy\, dz \; e^{-\frac{\beta}{2m}\left[\left(p_x - \frac{qyB_0}{2c}\right)^2 + \left(p_y + \frac{qxB_0}{2c}\right)^2 + p_z^2\right]}$$

change variables to: $u_x \equiv p_x - \frac{qyB_0}{2c}$; $u_y = p_y + \frac{qxB_0}{2c}$

$$\int_{-\infty}^{\infty} du_x \int_{-\infty}^{\infty} du_y \int_{-\infty}^{\infty} dp_z \int dx\, dy\, dz \; \underbrace{e^{-\frac{\beta}{2m}\left[u_x^2 + u_y^2 + u_z^2\right]}}_{\text{independent of } (x,y,z)}$$

$$V\left[\int_{-\infty}^{\infty} du_x \, e^{-\frac{\beta}{2m}u_x^2}\right]^3 = V\left(2\pi m k T\right)^{3/2}$$

$$\Rightarrow \quad Z_N = \frac{1}{N!}\frac{\left(2\pi m k T\right)^{3N/2}}{h^{3N}} V^N$$

is independent of B_0 ⇒ like a usual unmagnetized ideal gas

SM #1 (answer...)

a) Eq. of State is: $\bar{P} = \frac{1}{\beta}\frac{\partial}{\partial V} \ln Z_N = \frac{1}{\beta}\frac{\partial}{\partial V}\left[N \ln V\right]$

$$\Rightarrow \boxed{\bar{P} = \frac{NkT}{V}}$$ ideal gas Eq. of state

b) $$\boxed{\langle M \rangle = \frac{1}{\beta}\frac{\partial}{\partial B_0} \ln Z_N = 0}$$ which is Bohr's Thesis result.

c) $$\langle E^2 \rangle - \bar{E}^2 = \frac{\partial^2}{\partial \beta^2} \ln Z_N = -\frac{\partial}{\partial \beta} \bar{E}$$

with $$\bar{E} \equiv \langle E \rangle = -\frac{\partial}{\partial \beta} \ln Z_N = -\frac{\partial}{\partial \beta}\left[-\frac{3N}{2} \ln \beta \right]$$

$$\bar{E} = \frac{3N}{2}\frac{1}{\beta}$$

$$\Rightarrow \frac{\partial}{\partial \beta}\bar{E} = -\frac{3N}{2\beta^2} = -\frac{1}{\beta}\bar{E} = -\frac{2}{3N}\bar{E}^2$$

$$\Rightarrow \langle E^2 \rangle - \bar{E}^2 = \frac{2}{3N}\bar{E}^2$$

$$\Rightarrow \boxed{\sqrt{\frac{\langle E^2 \rangle - \bar{E}^2}{\bar{E}^2}} = \sqrt{\frac{2}{3N}}}$$

Prob 11.

Principle of superposition $\underset{\sim}{H} = \underset{\sim}{H}_2 - \underline{H}_1$

$\underline{H}_2$ = field of solid , $\underline{H}_1$ = field of hole

I_2, $\underline{j}$, I_1, $-\underline{j}$

$a := \frac{D}{2}$ radius of small hole

$\underline{I} = I_2 - I_1 = 9\pi a^2 j - \pi a^2 j = 8\pi a^2 j$

$\Rightarrow j = \frac{I}{8\pi a^2}$

$I_1 = \pi a^2 j = \frac{I}{8}$

$I_2 = 9\pi a^2 j = \frac{9}{8} I$

current flows out of paper plane

Ampere's law

$H_{2x} = -\frac{I \cdot y}{16\pi a^2}$, $H_{2y} = \frac{Ix}{16\pi a^2}$ inside $(r \le 3a)$

$(r = \sqrt{x^2+y^2})$

$H_{2x} = -\frac{9Iy}{16\pi(x^2+y^2)}$, $H_{2y} = \frac{9Ix}{16\pi(x^2+y^2)}$

outside $(r > 3a)$

$$H_{1x} = -\frac{Iy}{16\pi a^2}, \quad H_{1y} = \frac{I(x-a)}{16\pi a^2} \quad (r_1 \le a)$$

where $r_1 = \sqrt{(x-a)^2+y^2}$ considers offset

$$H_{1x} = -\frac{Iy}{16\pi[(x-a)^2+y^2]}, \quad H_{1y} = \frac{I(x-a)}{16\pi[(x-a)^2+y^2]} \quad (r_1 > a)$$

On the plane P, $H_{2x} = H_{1x} = 0$

$\Rightarrow H_x = 0$

$H_y = H_{2y} - H_{1y}$

$\Rightarrow$ inside hole $(0 < x < 2a)$

$$H_y = \frac{Ix}{16\pi a^2} - \frac{I(x-a)}{16\pi a^2} = \frac{Ia}{16\pi a^2} = \frac{I}{16\pi a}$$

inside solid $(2a \le x \le 3a$ or $-3a \le x \le 0)$

$$H_y = \frac{Ix}{16\pi a^2} - \frac{I(x-a)}{16\pi[(x-a)^2+y^2]} = \frac{I(x^2-ax-a^2)}{16\pi a^2(x-a)}$$

outside cylinder

$(|x| > 3a)$

$$H_y = \frac{9Ix}{16\pi(x^2+y^2)} - \frac{I(x-a)}{16\pi[(x-a)^2+y^2]} = \frac{(8x-9a)I}{16\pi x(x-a)}$$

The magnetic field at all points inside the hole $(r_1 \le a)$

$$H_x = -\frac{Iy}{16\pi a^2} + \frac{Iy}{16\pi a^2} = 0$$

$$H_y = \frac{Ix}{16\pi a^2} - \frac{I(x-a)}{16\pi a^2} = \frac{I}{16\pi a}$$

$\Rightarrow$ uniform and along y-axis

Q.12

A pulsar emits bursts of radio waves which are observed from the Earth at two different frequencies, ω_1 and ω_2. An astronomer notes that the arrival time of the bursts is delayed at the lower frequency, that is, the pulse at ω_1 arrives after the pulse at ω_2. The delay, τ is due to dispersion in the interstellar medium. Assume this medium consists of ionized hydrogen (called the dilute plasma), m_e is the mass of the electron and N the number of electrons per unit volume.

(a). Find the index of refraction of the dilute plasma.

(b). Find the distance from the pulsar to the Earth.

Solution

(a) For an electron in the dilute plasma, we have $m_e\ddot{\vec{x}} = -e\vec{E}$ and $\vec{E} = \vec{E}_0 e^{-i\omega t}$

Solving the above equaiton, we get the dipole moment of the electron,

$$\vec{p} = -e\vec{x} = -\frac{(e^2/m)}{\omega^2}\vec{E}$$

The polarization is given by

$$\vec{P} = -\frac{(Ne^2/m)}{\omega^2}\vec{E}$$

As $\vec{P} = \varepsilon_0 \chi_e \vec{E}$, we have $\varepsilon_r = 1 + \chi_e = 1 - \frac{\omega_p^2}{\omega^2}$ where $\omega_p = e\sqrt{\frac{N}{m\varepsilon_0}}$.

The index of refraction of the dilute plasma is $n = \sqrt{\varepsilon_r} = \sqrt{1 - \frac{\omega_p^2}{\omega^2}}$

(b) The pulse the pulsar emits is a wave packet that travels to the Earth. As such, it travels at the **group** velocity, not the **phase** velocity.

$$k = \frac{\omega}{c}n = \frac{\omega}{c}\sqrt{1 - \frac{\omega_p^2}{\omega^2}}$$

$$\frac{1}{v_g} = \frac{\partial k}{\partial \omega} = \frac{1}{c}\left[\frac{\omega}{2}\left(1 - \frac{\omega_p^2}{\omega^2}\right)^{-1/2}\left(\frac{2\omega_p^2}{\omega^3}\right) + \left(1 - \frac{\omega_p^2}{\omega^2}\right)^{1/2}\right]$$

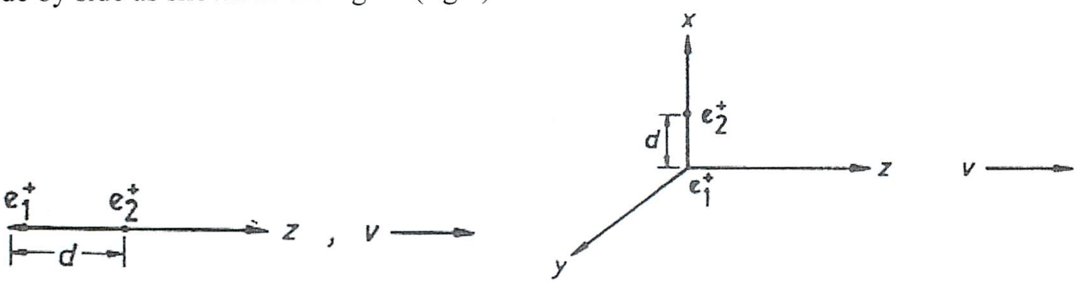
$$= \cdots = \frac{1}{c}\left(1 - \frac{\omega_p^2}{\omega^2}\right)^{-1/2}$$

$\tau = t_1 - t_2 = l\left(\frac{1}{v_{g1}} - \frac{1}{v_{g2}}\right) = \frac{l\omega_p^2}{2c}\left(\frac{1}{\omega_1^2} - \frac{1}{\omega_2^2}\right)$ where l is the distance from the pulsar to the Earth.

Q.13

(a) Consider two positrons in a beam at SLAC. The beam has energy of about 50 GeV ($\gamma \approx 10^5$). In the beam (rest) frame, they are separated by a distance d, and positron e_2^+ is traveling directly ahead of e_1^+ in the Z-axis, shown in the figure (left) below. Write down $\vec{E}$, $\vec{B}$, the Lorentz force $\vec{F}$ and the acceleration $\vec{a}$ on e_1^+ exerted by e_2^+. Do this in both the rest and laboratory frames.

(b) The problem is the same as in part (a) except this time the two positrons are traveling side by side as shown in the figure (right) below.



Two positrons separated by a distance of d travel with a velocity of v in the Z axis.

Solution

(a) Let K' and K the beam rest and laboratory frames, respectively.

In frame K',

$$\vec{E}' = -\frac{1}{4\pi\varepsilon_0}\frac{e}{d^2}\hat{z} \qquad \vec{B}' = 0 \qquad \vec{F}' = -\frac{1}{4\pi\varepsilon_0}\frac{e^2}{d^2}\hat{z} \qquad \vec{a}' = -\frac{1}{4\pi m\varepsilon_0}\frac{e^2}{d^2}\hat{z}$$

In frame K,

$$\vec{E} = \vec{E}' = -\frac{1}{4\pi\varepsilon_0}\frac{e}{d^2}\hat{z} \qquad \vec{B} = 0 \qquad \vec{F} = e\vec{E} = \vec{F}'$$

$$\vec{a} = c\frac{d\beta}{dt} = \frac{1}{m\gamma}\left(F - F\beta^2\right) = \frac{F}{m\gamma^3} = \frac{\vec{a}'}{\gamma^3}$$

In this case, the EM filed and the Lorentz force are the same in K and K'. Due to the relativistic effect the acceleration of e_1^+ in frame K is only $\frac{1}{\gamma^3}$ times that in the rest frame.

In other words, the force exerted by a neighboring collinear charge on a charge moving with high speed will be small.

(b) In frame K',

$$\vec{E}' = -\frac{1}{4\pi\varepsilon_0}\frac{e}{d^2}\hat{x} \qquad \vec{B}' = 0 \qquad \vec{F}' = -\frac{1}{4\pi\varepsilon_0}\frac{e^2}{d^2}\hat{x} \qquad \vec{a}' = -\frac{1}{4\pi m\varepsilon_0}\frac{e^2}{d^2}\hat{x}$$

In frame K,

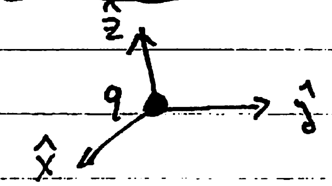
$$\vec{E} = \gamma\vec{E}' \qquad \vec{B} = \gamma\frac{v}{c^2}E'\,\hat{y} \qquad \vec{F} = \frac{\vec{F}'}{\gamma}$$

Q13, p2

$$\vec{a} = \frac{F}{m\gamma} = \frac{\vec{a}'}{\gamma^2}$$

Q 14.

E+M #1 (answer).

a) 

$\nabla\cdot\underline{E} = 4\pi(\rho_{ext} + \rho_p)$ is Poisson's Eq

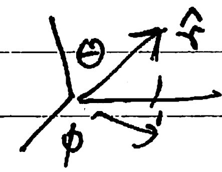
where $\rho_{ext} = q\,\delta(\underline{r})$; ρ_p is induced polarization chg

using displacement vector $\underline{D} = \overleftrightarrow{\epsilon}\cdot\underline{E}$ transforms Poisson's Eq into: $\nabla\cdot\underline{D} = 4\pi q\,\delta(\underline{r})$

solvable by Gauss' law $\oint$: $\underline{D} = \dfrac{q}{r^2}\hat{r}$

with $r = (x^2+y^2+z^2)$; $\hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$

in spherical coordinates



or $$\hat{r} = \frac{x}{r}\hat{x} + \frac{y}{r}\hat{y} + \frac{z}{r}\hat{z}$$

$$\Rightarrow \quad \underline{D} = \frac{q}{r^3}\left[x\,\hat{x} + y\,\hat{y} + z\,\hat{z}\right]$$

$$+ \quad \overleftrightarrow{\epsilon}\cdot\underline{E} = \epsilon_\perp E_x\,\hat{x} + \epsilon_\perp E_y\,\hat{y} + \epsilon_\parallel E_z\,\hat{z}$$

$$\Rightarrow \quad E_x = \frac{q}{r^3\epsilon_\perp}x, \quad E_y = \frac{q}{r^3\epsilon_\perp}y, \quad E_z = \frac{q}{r^3\epsilon_\parallel}z$$

+ the magnitude is: $|\underline{E}| = \sqrt{E_x^2 + E_y^2 + E_z^2}$

$$\boxed{|\underline{E}| = \frac{q}{r^3}\left[\frac{x^2+y^2}{\epsilon_\perp^2} + \frac{z^2}{\epsilon_\parallel^2}\right]^{1/2}}$$

E+M #1 (answer/cont..)

b) For $r \neq 0$ all the charge density is due to polarization of the dielectric, i.e., $\nabla \cdot \underline{E} = 4\pi \rho_p \Rightarrow \rho_p = \frac{1}{4\pi} \nabla \cdot \underline{E}$

but $\nabla \cdot \underline{E} = \partial_x E_x + \partial_y E_y + \partial_z E_z$

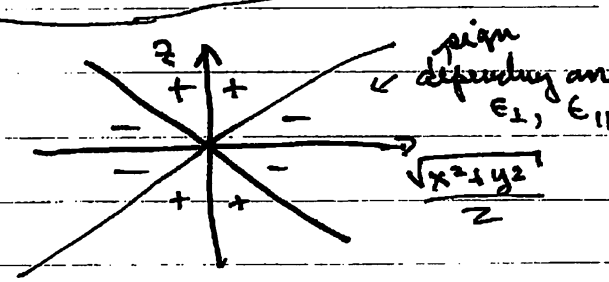
and
$$\partial_x E_x = \frac{q}{\epsilon_\perp} \frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) = \frac{q}{\epsilon_\perp}\left[\frac{1}{r^3} - \frac{3x^2}{r^5}\right]$$
$$\partial_y E_y = \frac{q}{\epsilon_\perp} \frac{\partial}{\partial y}\left(\frac{y}{r^3}\right) = \frac{q}{\epsilon_\perp}\left[\frac{1}{r^3} - \frac{3y^2}{r^5}\right]$$
$$\partial_z E_z = \frac{q}{\epsilon_\parallel} \frac{\partial}{\partial z}\left(\frac{z}{r^3}\right) = \frac{q}{\epsilon_\perp}\left[\frac{1}{r^3} - \frac{3z^2}{r^5}\right]$$

$$\Rightarrow \nabla \cdot \underline{E} = \frac{q}{r^5}\left\{\frac{(2z^2 - x^2 - y^2)}{\epsilon_\perp} + \frac{x^2 + y^2 - 2z^2}{\epsilon_\parallel}\right\}$$

$$\nabla \cdot \underline{E} = \frac{q}{r^5}\left[2z^2 - (x^2+y^2)\right]\left(\frac{1}{\epsilon_\perp} - \frac{1}{\epsilon_\parallel}\right)$$

$$\Rightarrow \boxed{\rho_p(x,y,z) = \frac{q}{4\pi r^5}\left[2z^2 - (x^2+y^2)\right]\left(\frac{1}{\epsilon_\perp} - \frac{1}{\epsilon_\parallel}\right)}$$

shows charge cones



c) Total electrical energy density is $u_E = \frac{1}{2}\underline{E} \cdot \underline{D} = \frac{1}{2}\underline{E} \cdot \overleftrightarrow{\epsilon} \cdot \underline{E}$

but $\overleftrightarrow{\epsilon} \cdot \underline{E} = \epsilon_\perp E_x \hat{x} + \epsilon_\perp E_y \hat{y} + \epsilon_\parallel E_z \hat{z}$

$\therefore \underline{E} \cdot \overleftrightarrow{\epsilon} \cdot \underline{E} = \epsilon_\perp E_x^2 + \epsilon_\perp E_y^2 + \epsilon_\parallel E_z^2$

III.

E+M #1 (answer) (cont.)

$$u_E = \frac{1}{2}\left[\epsilon_\perp \frac{q^2}{r^6\epsilon_\perp^2}x^2 + \frac{q^2}{r^6\epsilon_\perp^2}y^2 + \frac{q^2}{r^6\epsilon_\parallel^2}z^2\right]$$

$$\boxed{u_E = \frac{q^2}{2r^6}\left[\frac{x^2+y^2}{\epsilon_\perp} + \frac{z^2}{\epsilon_\parallel}\right]}$$