CLASSICAL MECHANICS 220

Final Exam – Fall 2011

Monday 5 December 2011, at 3 - 6pm, in PAB 2434

- Please write clearly;
- Print your name on every page used, including this one;
- Make clear which question you are answering on each page;
- All five questions below are independent from one another.
- No books, notes, computers, or calculators are allowed during the exam;
- Please turn off cell-phones, iPhones, iPods, iPads, Kindles, and other electronic devices.

Grades

Q1.

Q2.

Q3.

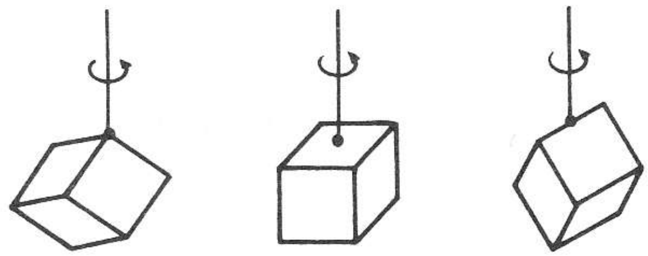
Q4.

Q5.

Total /60

QUESTION 1 [12 points]

A torsion pendulum consists of a vertical wire attached to a mass which may rotate about the vertical direction. Consider three torsion pendulums which consist of identical wires from which identical homogeneous solid cubes are suspended. One cube is suspended from a vertex, one from the center of an edge, and one from the center of a face, as shown in the figure. What are the ratios of the periods of the three pendulums ? Justify your answer.



QUESTION 2 [12 points]

A non-relativistic particle with mass m and electric charge e moves in a two-dimensional plane (with Cartesian coordinates x, y), under the influence of a constant uniform magnetic field $\mathbf{B} = (0, 0, B)$, and an <u>*inverted*</u> harmonic oscillator potential,

$$V(x, y) = -\frac{1}{2} m\omega^2 (x^2 + y^2) \tag{0.1}$$

(a) Write down the Lagrangian for this system in the xy plane;

(b) Derive the corresponding Euler-Lagrange equations;

(c) Solve the Euler-Lagrange equations;

(d) Discuss stability of motion near the point $(x, y) = (0, 0)$ as a function of m, e, ω, B.

QUESTION 3 [12 points]

A relativistic particle with rest mass m, and electric charge e moves in an electro-magnetic field which is constant in space and in time, and whose components with respect to an inertial frame $\mathcal{R}$ are given by $\mathbf{E} = (E, 0, 0)$ and $\mathbf{B} = (0, 0, B)$.

(a) State the differential equation for the particle's velocity 4-vector u^μ with respect to $\mathcal{R}$, and work out the equations, component-by-component.

(b) Show that the solutions to this equation are linear superpositions of exponentials, and determine the corresponding exponents.

(c) Determine the conditions on E and B under which all components of u^μ are bounded along every trajectory. Is this condition Lorentz-invariant ?

QUESTION 4 [12 points]

The sine-Gordon model is a field theory in 1 space and 1 time dimensions with a single scalar field $\phi(t,x)$, governed by the following action,

$$S[\phi] = \int dtdx \left(\frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}(\partial_x \phi)^2 - m^2(1-\cos\phi) \right) \tag{0.2}$$

We shall work with units in which the speed of light is set to 1. Also, m is a real constant.

(a) Use the variational principle to obtain the corresponding Euler-Lagrange equation, and give the expression for the total energy E of a general field $\phi(t,x)$.

(b) Show that the Euler-Lagrange equation of (a) admits *solitons*: solutions of the form $\phi(t,x) = f(y)$, with $y = \lambda(v)(x - vt)$, for arbitrary velocity v, such that $\cos f(\pm\infty) = 0$, and $f(+\infty) \neq f(-\infty)$. Show that it is possible to choose $\lambda(v)$ such that f (as a function) is independent of v; determine this $\lambda(v)$, and the corresponding function f.

(c) Derive the relation between the total energy E of the soliton and its velocity v, and show that this relation is the relativistic one. Derive the mass of the soliton.

QUESTION 5 [12 points]

Two identical wheels of radius a and mass m are mounted vertically on the ends of a common horizontal axle (of zero mass and length $2b$) such that the wheels rotate independently. The whole system rolls freely without slipping or sliding on the horizontal plane $z = 0$.
A figure of the system is given on the next page to illustrate the set-up.

(a) Show that the coordinates x, y, θ and $\varphi = (\varphi_1 + \varphi_2)/2$, indicated in figure 1, may be used as suitable generalized coordinates in which all holonomic constraints, and any integrable non-holonomic constraints, have been eliminated.

(b) Exhibit the non-holonomic constraint(s) that remain on x, y, θ, φ, if any.

(c) Write down the Lagrangian when no external forces, other than those associated with the constraints, operate on the system.

(d) With the method of Lagrange multipliers, derive the Euler-Lagrange equations.

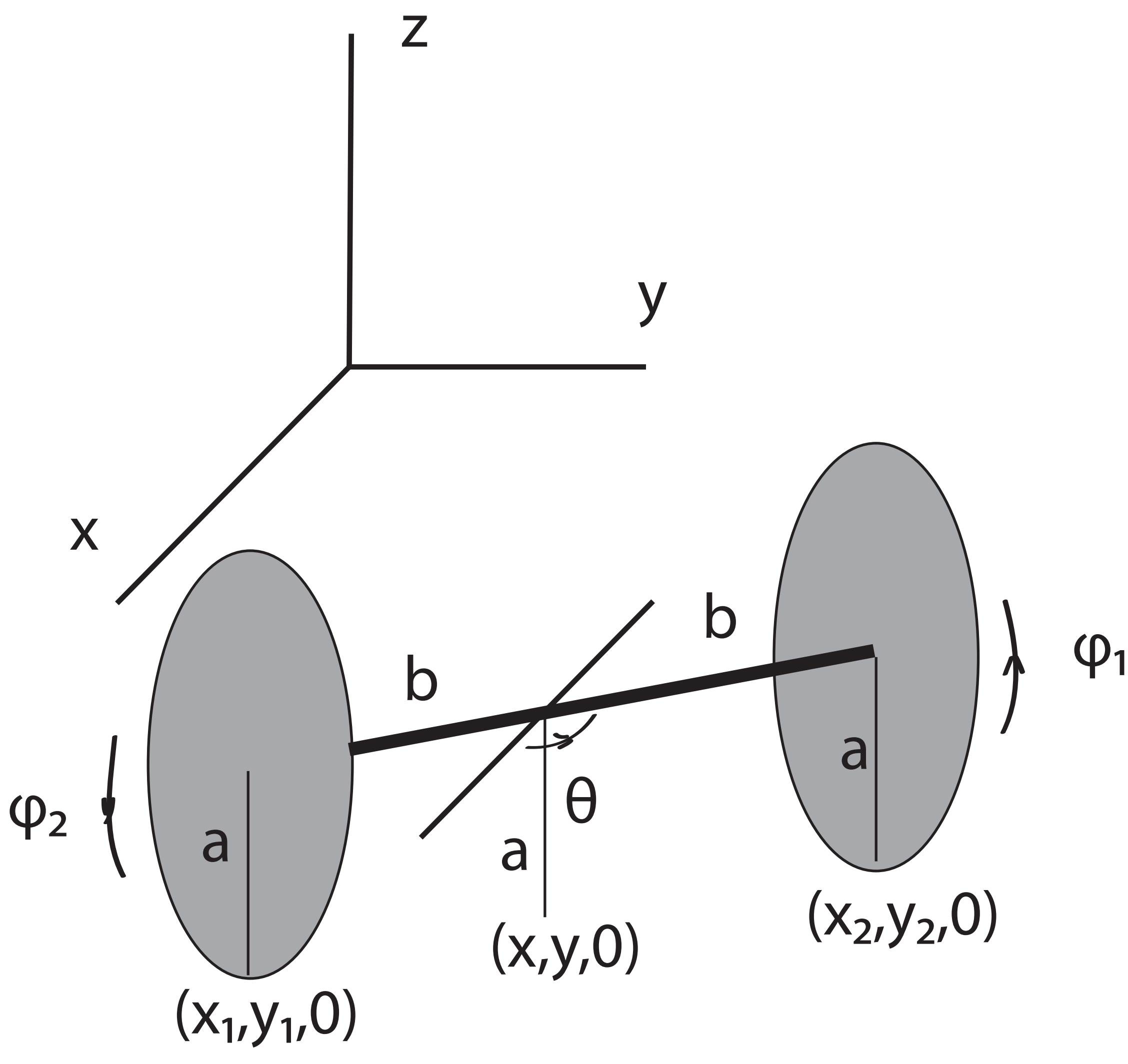
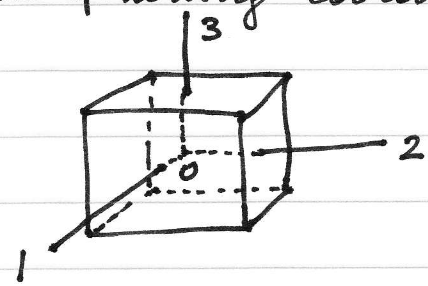


Figure 1: Two vertical wheels mounted on a horizontal axle.

Mechanics 220
Final Exam
Fall 2011.
Solutions.

(1). Consider a homogeneous solid cube with respect to the following coordinate system:



(a) The body of the cube is now symmetrical under reflections of any coordinate axis: $x_1 \to -x_1$, $x_2 \to -x_2$, $x_3 \to -x_3$.
The moment of inertia tensor thus has vanishing off-diagonal components, since each of these is odd under two reflections.

(b) The diagonal entries of the inertia tensor are equal to one another as the figure above is clearly symmetrical under interchange of the three axes.

(c) Since the inertia tensor is proportional to the identity, and each rotation of the problem passes through the C.M. of the cube 0:
<u>the periods of all three rotations coincide.</u>

(2) The general Lagrangian for a charged particle in an e.m. field is (in 2-dimensions).

$$L = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) + e A_\mu(x)\, \dot{x}^\mu - V(x, y).$$

We choose a gauge for constant magnetic field B:

$$\begin{cases} A_2 = \frac{1}{2} Bx \\ A_1 = -\frac{1}{2} By. \end{cases}$$

(a) Hence our Lagrangian is given by,

$$L = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) + \frac{eB}{2}(x\dot{y} - y\dot{x}) + \frac{1}{2} m\omega^2(x^2 + y^2).$$

(b) $$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} - \frac{1}{2} eBy$$

$$p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y} + \frac{1}{2} eBx.$$

$$\frac{\partial L}{\partial x} = \frac{eB}{2}\dot{y} + m\omega^2 x$$

$$\frac{\partial L}{\partial y} = -\frac{eB}{2}\dot{x} + m\omega^2 y.$$

E-L. eqs:
$$\begin{cases} m\ddot{x} - eB\dot{y} - m\omega^2 x = 0 \\ m\ddot{y} + eB\dot{x} - m\omega^2 y = 0. \end{cases}$$

(c) The eqs form a linear system, with constant coefficients, of ordinary differential equations.

$$\begin{pmatrix} \frac{d^2}{dt^2} - \omega^2 & -\frac{eB}{m}\frac{d}{dt} \\ \frac{eB}{m}\frac{d}{dt} & \frac{d^2}{dt^2} - \omega^2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 .$$

Solutions are generally exponentials of t.

$$\begin{pmatrix} x \\ y \end{pmatrix}(t) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} e^{\lambda t}$$

$$\begin{pmatrix} \lambda^2 - \omega^2 & -\frac{eB}{m}\lambda \\ \frac{eB}{m}\lambda & \lambda^2 - \omega^2 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = 0 .$$

Vanishing determinant:

$$(\lambda^2 - \omega^2)^2 + \frac{e^2B^2}{m^2}\lambda^2 = 0$$

$$\Rightarrow \quad \left(\lambda^2 - \omega^2 + \frac{ieB}{m}\lambda\right)\left(\lambda^2 - \omega^2 - \frac{ieB}{m}\lambda\right) = 0$$

First factor: vanishes when

$$\left(\lambda + \frac{ieB}{2m}\right)^2 = \omega^2 - \frac{e^2B^2}{4m^2} .$$

$$\lambda = -\frac{ieB}{2m} \pm \sqrt{\omega^2 - \frac{e^2B^2}{4m^2}} .$$

Second factor vanishes when

$$\lambda = +\frac{ieB}{2m} \pm \sqrt{\omega^2 - \frac{e^2B^2}{4m^2}} .$$

(d). $\omega^2 > \frac{e^2B^2}{4m^2}$: unstable $\qquad \omega^2 \leq \frac{e^2B^2}{4m^2}$: stable.

(3) Convenient to work with 4-vector notation.

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E/c & 0 & 0 \\ E/c & 0 & B & 0 \\ 0 & -B & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(a) $$m\frac{du^\mu}{d\tau} = e\,F^{\mu\nu}\,u_\nu = e\,\eta^{\mu\kappa}\,F_{\kappa\nu}\,u^\nu$$

$$m\frac{du^0}{d\tau} = \frac{eE}{c}u^1$$

$$m\frac{du^1}{d\tau} = \frac{eE}{c}u^0 + eB\,u^2$$

$$m\frac{du^2}{d\tau} = -eB\,u^1$$

$$m\frac{du^3}{d\tau} = 0\,.$$

(b). u^3 = constant; other eq. const. coeff ⇒ exp. sols.

$$\begin{pmatrix} u^0 \\ u^1 \\ u^2 \end{pmatrix} = \begin{pmatrix} \alpha^0 \\ \alpha^1 \\ \alpha^2 \end{pmatrix} e^{\lambda\tau}$$

$$\begin{pmatrix} m\frac{d}{d\tau} & -eE/c & 0 \\ -eE/c & m\frac{d}{d\tau} & -eB \\ 0 & eB & m\frac{d}{d\tau} \end{pmatrix}\begin{pmatrix} \alpha^0 \\ \alpha^1 \\ \alpha^2 \end{pmatrix} e^{\lambda\tau} = 0\,.$$

$$\det \begin{pmatrix} \lambda m & -eE/c & 0 \\ -eE/c & \lambda m & -eB \\ 0 & eB & \lambda m \end{pmatrix} = 0$$

$$m\lambda\left(m^2\lambda^2 + e^2B^2\right) + eE/c \begin{vmatrix} -eE/c & 0 \\ eB & \lambda m \end{vmatrix} = 0 .$$

$$m\lambda\left(m^2\lambda^2 + B^2\right) - m\lambda \frac{e^2E^2}{c^2} = 0$$

$$m\lambda\left((m\lambda)^2 + B^2 - \frac{E^2}{c^2}\right) = 0 \qquad \lambda = 0,\ m\lambda = \pm\sqrt{\frac{E^2}{c^2} - B^2}$$

(c) $\lambda = 0$: $\begin{pmatrix} \alpha^0 \\ \alpha^1 \\ \alpha^2 \end{pmatrix} = \begin{pmatrix} \alpha B \\ 0 \\ -\alpha E/c \end{pmatrix}$ α constant.

this is bounded.

$\lambda = \pm\sqrt{\frac{E^2}{c^2} - B^2}$: If $E^2 > B^2c^2$: unbounded.
If $E^2 \leq B^2c^2$: bounded.

The condition is Lorentz invariant, since

$$-\frac{1}{2}\left(\frac{\vec{E}^2}{c^2} - \vec{B}^2\right) = F_{\mu\nu}F^{\mu\nu} .$$

is Lorentz-invariant.

(4)

$$S[\phi] = \int dt\, dx \left(\frac{1}{2} (\partial_t \phi)^2 - \frac{1}{2} (\partial_x \phi)^2 - m^2 (1 - \cos\phi) \right)$$

(a). $\delta S[\phi] = \int dt\, dx\, \delta\phi \left(-\partial_t^2 \phi + \partial_x^2 \phi - m^2 \sin\phi \right)$

Then field eq is $\quad \partial_t^2 \phi - \partial_x^2 \phi + m^2 \sin\phi = 0.$

The Lagrangian has the form $L = T - V$, so the energy is simply

$$H = E = \int dx \left(\frac{1}{2} (\partial_t \phi)^2 + \frac{1}{2} (\partial_x \phi)^2 + m^2 (1 - \cos\phi) \right).$$

(b) $\phi(t, x) = f(\lambda(x - vt)) \qquad \lambda, v$ constant.

$\partial_t^2 \phi = \lambda^2 v^2 f'' \qquad f''$ = double derivative
$\partial_x^2 \phi = \lambda^2 f'' \qquad$ with respect to argument.

$$-\lambda^2 (1 - v^2) f'' + m^2 \sin f = 0$$

Choosing $\lambda^2 (1 - v^2) = 1$, or $\lambda = \gamma = (1 - v^2)^{-1/2}$, the equation for f is independent of v:

$$f'' - m^2 \sin f = 0$$

$$\Rightarrow \quad f' f'' - m^2 \sin f\, f' = 0 \qquad f' \neq 0.$$

$$\frac{1}{2} (f')^2 + m^2 \cos f = \text{constant}.$$

Require soliton behavior; $\cos f \to 1$ as $x \to \pm\infty$, so that also $f' \to 0$ in that limit.

Hence the equation becomes:

$$(f')^2 = 2m^2(1-\cos f) = 4m^2 \sin^2\frac{f}{2}.$$

$$\frac{f'}{2} = \pm m \sin\frac{f}{2}$$

Rationalize in terms of $g = tg\frac{f}{4}$

$$\sin\frac{f}{2} = \frac{2g}{1+g^2} \qquad \left(\frac{f}{2}\right)' = \frac{2g'}{1+g^2}.$$

In terms of g, the eq. becomes

$$2g' = \pm m\, 2g \qquad g(x) = e^{\pm my}$$

$$\Rightarrow \quad tg\frac{f}{4} = e^{\pm my}$$

(c) To derive the total energy, compute

$$tg\frac{\phi}{4} = e^{\pm m\gamma(x - vt)}$$

$$(\partial_t\phi)^2 = \gamma^2 v^2 (f')^2$$
$$(\partial_x\phi)^2 = \gamma^2 (f')^2$$
$$m^2(1-\cos\phi) = 2m^2\sin^2\frac{f}{2} = \frac{1}{2}(f')^2$$
$$= \frac{1}{2}\left((\partial_x\phi)^2 - (\partial_t\phi)^2\right).$$

Energy simplifies:

$$E = \int_{-\infty}^{\infty} dx\, (\partial_x\phi)^2$$

Computing this out:

$$E = \gamma^2 \int_{-\infty}^{\infty} dx \; f'(\gamma(x-vt))^2$$

$$= \gamma \int_{-\infty}^{\infty} dy \; f'(y)^2$$

$$= 16\gamma \int_{-\infty}^{\infty} dy \; \frac{(g')^2}{(1+g^2)^2}$$

$$= 16\gamma \int_{-\infty}^{\infty} dy \; \frac{m^2 e^{\pm 2my}}{(1+e^{\pm 2my})^2}$$

$$= 8m\gamma \int_{-\infty}^{\infty} \frac{(\pm)\, d\left(e^{\pm 2my}\right)}{(1+e^{\pm 2my})^2}.$$

$$E = 8m\gamma$$

Precisely relativistic formula for energy versus mass & velocity, with mass of soliton M

$$M = 8m.$$

5. The holonomic constraints are solved by

(a)&(b) $$x_1 = x - b\cos\theta \qquad x_2 = x + b\cos\theta$$
$$y_1 = y - b\sin\theta \qquad y_2 = y + b\sin\theta .$$

The non-holonomic constraints are

$$\dot{x}_1 = a\sin\theta\,\dot{\varphi}_1 \qquad \dot{x}_2 = a\sin\theta\,\dot{\varphi}_2$$
$$\dot{y}_1 = -a\cos\theta\,\dot{\varphi}_1 \qquad \dot{y}_2 = -a\cos\theta\,\dot{\varphi}_2$$

Working with coordinates $x, y, \theta, \varphi_1, \varphi_2$ is achieved by eliminating x_1, x_2, y_1, y_2 using the holonomic constraints:

$$\dot{x} + b\sin\theta\,\dot{\theta} = a\sin\theta\,\dot{\varphi}_1$$

$$\dot{x} - b\sin\theta\,\dot{\theta} = a\sin\theta\,\dot{\varphi}_2$$

$$\dot{y} - b\cos\theta\,\dot{\theta} = -a\cos\theta\,\dot{\varphi}_1$$

$$\dot{y} + b\cos\theta\,\dot{\theta} = -a\cos\theta\,\dot{\varphi}_2$$

Subtracting to eliminate $\dot{x}$ and $\dot{y}$ gives

$$2b\sin\theta\,\dot{\theta} = a\sin\theta\,(\dot{\varphi}_1 - \dot{\varphi}_2)$$

$$2b\cos\theta\,\dot{\theta} = a\cos\theta\,(\dot{\varphi}_1 - \dot{\varphi}_2).$$

Equations are prop. to same $2b\dot{\theta} = a(\dot{\varphi}_1 - \dot{\varphi}_2)$.

But this constraint is integrable:

$$2b\theta = a(\varphi_1 - \varphi_2)$$

(A possible additive integration constant may be absorbed into the def. of φ_1 and φ_2.). This leaves as independent variables $x, y, \theta, \varphi = (\varphi_1+\varphi_2)/2$.
The remaining non-holonomic constraints are:

$$\begin{cases} \dot{x} = a \sin\theta \, \dot{\varphi} \\ \dot{y} = -a \cos\theta \, \dot{\varphi} \end{cases}$$

(c). The Lagrangian is

$$L = \frac{1}{2} m (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m (\dot{x}_2^2 + \dot{y}_2^2)$$

$$+ \frac{1}{2} I_1 (\dot{\varphi}_1^2 + \dot{\varphi}_2^2) + \frac{1}{2} I_2 \dot{\theta}^2 + \frac{1}{2} I_2 \dot{\theta}^2$$

$$I_1 = \frac{1}{2} m a^2 \qquad\qquad I_2 = \frac{1}{4} m a^2$$

In terms of coordinates $x, y, \theta, \varphi_1, \varphi_2$

$$L = m(\dot{x}^2 + \dot{y}^2) + m^2 b^2 \dot{\theta}^2 + I_2 \dot{\theta}^2 + \frac{1}{2} I_1 (\dot{\varphi}_1^2 + \dot{\varphi}_2^2)$$

$$\varphi_1 = \varphi + \frac{b}{a}\theta \qquad\qquad \dot{\varphi}_1^2 + \dot{\varphi}_2^2 = 2\dot{\varphi}^2 + 2\frac{b^2}{a^2}\dot{\theta}^2$$

$$\varphi_2 = \varphi - \frac{b}{a}\theta \qquad\qquad I \equiv m^2 b^2 + I_2 + I_1 \frac{b^2}{a^2}$$

$$L = m(\dot{x}^2 + \dot{y}^2) + I\dot{\theta}^2 + I_1 \dot{\varphi}^2$$

(d). The remaining constraints are of the form linear and homogeneous in velocities:

$$C_x = \dot{x} - a \sin\theta\, \dot{\varphi}$$

$$C_y = \dot{y} + a \cos\theta\, \dot{\varphi}$$

The Euler-Lagrange equations take the form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \mu_x \frac{\partial C_x}{\partial \dot{q}_i} + \mu_y \frac{\partial C_y}{\partial \dot{q}_i}$$

$$\frac{\partial L}{\partial \dot{x}} = 2m\dot{x} \qquad \frac{\partial L}{\partial \dot{\theta}} = 2I\dot{\theta}$$

$$\frac{\partial L}{\partial \dot{y}} = 2m\dot{y} \qquad \frac{\partial L}{\partial \dot{\varphi}} = 2I_1\dot{\varphi}$$

$$\begin{cases} 2m\ddot{x} = \mu_x \\ 2m\ddot{y} = \mu_y \\ 2I_1\ddot{\varphi} = -a\sin\theta\,\mu_x + a\cos\theta\,\mu_y \\ 2I\ddot{\theta} = 0 \end{cases}$$

Together with the constraints $C_x = C_y = 0$.

The system is easily solved. Compute the time derivatives of the constraints:

$$\begin{cases} \ddot{x} = a \sin\theta\, \ddot{\varphi} + a\cos\theta\, \dot{\theta}\, \dot{\varphi} \\ \ddot{y} = -a\cos\theta\, \ddot{\varphi} + a\sin\theta\, \dot{\theta}\, \dot{\varphi} \end{cases}$$

Also, we have from the last eq of motion:

$$\dot{\theta} = \text{constant}.$$

Eliminating $\ddot{x}$, $\ddot{y}$ and μ_x, μ_y in the φ eq:

$$2 I_1 \ddot{\varphi} = -2ma\sin\theta\left(a\sin\theta\,\ddot{\varphi} + a\cos\theta\,\dot{\theta}\,\dot{\varphi}\right)$$
$$+ 2ma\cos\theta\left(-a\cos\theta\,\ddot{\varphi} + a\sin\theta\,\dot{\theta}\,\dot{\varphi}\right)$$

The $\dot{\theta}\dot{\varphi}$ parts cancel, and the remainder reads:

$$(2I_1 + 2ma^2)\,\ddot{\varphi} = 0$$

Since $I_1 + ma^2 > 0$, we have

$$\dot{\varphi} = \text{constant}.$$

The general solution is then

$$\theta(t) = \theta_0 + \omega t \qquad \omega, \Omega \text{ constant}.$$
$$\varphi(t) = \varphi_0 + \Omega t$$

$$\dot{x} = a\sin(\omega t + \theta_0)\,\Omega \qquad x(t) = x_0 - \frac{a\Omega}{\omega}\cos(\omega t + \theta_0)$$
$$\dot{y} = -a\cos(\omega t + \theta_0)\,\Omega \qquad y(t) = y_0 - \frac{a\Omega}{\omega}\sin(\omega t + \theta_0)$$